Test 1-1

Solutions to Test 1: MA 575

1.(i) As a set $A = \{x_j\}$ is bdd so lub A exists. In fact lub A = $\limsup_j x_j$. Set x_∞ = lub A. The for any $\varepsilon > 0$ $\exists$ n_ε s.t. $|x_\infty - x_{n_\varepsilon}| < \varepsilon$. Since $\{x_j\}$ is monotone nondecr. if $m > n_\varepsilon$
$x_\infty - x_m = (x_\infty - x_{n_\varepsilon}) + (x_{n_\varepsilon} - x_m) < x_\infty - x_{n_\varepsilon}$ (since $x_{n_\varepsilon} \leq x_m$)
so $\forall\, m > n_\varepsilon$ $|x_\infty - x_m| < \varepsilon$ proving $x_j \to x_\infty$. ∎

(ii) $A \subset \mathbb{R}$ bdd above & closed. lub $A = a_\infty$ exists. Since a_∞ is the lub A if $x < a_\infty$ then x isn't an UB of A. So for each $\varepsilon > 0$ $\exists$ $a_\varepsilon \in A$ s.t. $N_\varepsilon(a_\infty)$ contains a_ε ($a_\varepsilon \neq a_\infty$). This proves a_∞ is a limit pt. of A and as A is closed, $a_\infty \in A$. (One can also construct a monotone incr. seq. a_j with $|a_j - a_\infty| < 1/\varepsilon$ so $a_j \to a_\infty$.)

2(i) A series $\sum_{j=1}^{\infty} z_j$ converges if the sequence of partial sums $S_n = \sum_{j=1}^{n} z_j$ converges.

(ii) Suppose $\sum_{j=1}^{\infty} a_j$ converges, then the seq of partial sums $s_n = \sum_{j=1}^{n} a_j$ is Cauchy so for $\varepsilon > 0$ $\exists$ N_ε s.t. $|s_n - s_m| = \left|\sum_{j=m+1}^{n} a_j\right| < \varepsilon$ (take $n > m$). Let $n = m+1$ so $|a_{m+1}| < \varepsilon$ for all $m > N_\varepsilon$. This implies $a_j \to 0$.

3(i) A seq $\{x_j\} \subset X$, metric sp., is Cauchy if $\forall\, \varepsilon > 0\, \exists\, N_\varepsilon$ s.t. $n, m > N_\varepsilon \Rightarrow d_X(x_n, x_m) < \varepsilon$.

(ii) Let $\{x_j\} \subset X$ be Cauchy. Suppose $\exists$ subseq $\{x_{n_j}\}_j$ that conv. to x_∞. Note $d(x_n, x_\infty) \leq d(x_n, x_{n_j}) + d(x_{n_j}, x_\infty)$. Given $\varepsilon > 0$ $\exists$ $N_\varepsilon^{(1)}$ s.t. $n, m > N_\varepsilon^{(1)} \Rightarrow d(x_n, x_m) < \varepsilon/2$, and $\exists$ $N_\varepsilon^{(2)}$ s.t. $j > N_\varepsilon^{(2)} \Rightarrow d(x_{n_j}, x_\infty) < \varepsilon/2$. Since $j \to n_j$ is incr. setting $N_\varepsilon = \max(N_\varepsilon^{(1)}, N_\varepsilon^{(2)})$, we find $n, j > N_\varepsilon \Rightarrow d(x_n, x_\infty) < \varepsilon$ so $x_n \to x_\infty$. ∎

(iii) A compact metric space X is complete. Pf: Let $\{x_j\}$ be a Cauchy seq with no convergent subseq. (if there was one then the seq would conv.) For every $x \in X$ $\exists \varepsilon > 0$, $\exists N_\varepsilon(x)$ contains at most finitely many pts (otherwise there would be a subseq of $\{x_j\}$ conv. to x). Take $\{N_\varepsilon(x)\}_{x \in X}$, any $\varepsilon > 0$. This is an open cover of X: $X \subset \bigcup_{x \in X} N_\varepsilon(x)$. Since X is compact, $\{N_\varepsilon(x_j)\}_{j=1}^{N}$ covers X. But this set contains at most finitely many pts of the Cauchy seq $\{x_j\}$. ∎

4. Defn: $\partial A = \bar{A^c} \cap \bar{A}$. Then $p \in \partial A \Leftrightarrow \forall \varepsilon > 0$ $N_\varepsilon(p) \cap A \neq \emptyset$ & $N_\varepsilon(p) \cap A^c \neq \emptyset$.
<u>Pf</u> $\Leftarrow$ $N_\varepsilon(p) \cap A \neq \emptyset \Rightarrow$ either $p \in A$ or p is a limit pt of A so $p \in \bar{A}$. Similarly, $p \in N_\varepsilon(p) \cap A^c$ $\forall \varepsilon > 0 \Rightarrow p \in \bar{A^c}$ so $p \in \partial A$.
$\Rightarrow$ $p \in \partial A \Rightarrow p \in \bar{A}$ so p is in A or a limit pt. $\Rightarrow$ $N_\varepsilon(p) \cap A \neq \emptyset$ $\forall \varepsilon > 0$. Similar for $p \in \bar{A^c}$. ∎