Test 1 - 1

(1a) $\{x_j\}$ bounded nondecr. $\Rightarrow$ the set is bounded above so $a = \text{lub}\{x_j\}$ exists.

For any $\varepsilon > 0$ $\exists$ $n_\varepsilon$ s.t. $a - \varepsilon \le x_{n_\varepsilon} \le a$ otherwise $a$ would not be an upper bound.

let $\varepsilon = 1/k$ $k = 1, 2, 3, \ldots$ and choose

$$x_k \text{ so } a - 1/k \le \cancel{x_k} \le a$$

Since $x_m \ge x_k$ $m \ge k$ and $x_m \le a$ we have

$$|x_m - a| < 1/k \quad \forall\, m > k.$$

This proves $x_m \to a$.

(1b) We show lub $A$ is a limit point.

Apply the same construction. For each $k$ $\exists$ $x_k$ so $a - 1/k \le x_k \le a$

This means each $N_\varepsilon(a)$ contains a point of $A$ so as $A$ is closed $a \in A$.

Test 1-2

(2a) $\Rightarrow$ Suppose $f: S \to T$ is continuous. Let $K \subset T$ be closed so $K^c \subset T$ is open.
We know $f^{-1}(K^c) \subset S$ is open since $f$ is cont.

Claim: $f^{-1}(K^c)^c = f^{-1}(K)$

Given this, $f^{-1}(K^c)^c$ is closed so $f^{-1}(K)$ is closed.

Pf of Claim: If $y \notin f^{-1}(K^c)$, $f(y) \notin K^c$
so $f(y) \in K \Rightarrow y \in f^{-1}(K)$
so $(f^{-1}(K^c))^c \subset f^{-1}(K)$
Conversely, if $w \in f^{-1}(K)$, $f(w) \in K$ so
$w \notin f^{-1}(K^c)$ so
$f^{-1}(K) \subset (f^{-1}(K^c))^c$ $\blacksquare$

$\Leftarrow$ Suppose $\forall K \subset T$ cld, $f^{-1}(K) \subset S$ is closed
Let $O \subset T$ be open so $O^c$ is closed
$f^{-1}(O^c) \subset S$ is closed and $(f^{-1}(O^c))^c \subset S$ is open. By the claim above
$(f^{-1}(O^c))^c = f^{-1}(O)$ is open. $\blacksquare$

(2b) $f: S \to T$, cont. & bijective & $S$ compact. $f^{-1}: T \to S$ exists
It is cont if $\forall K \subset S$ closed, $(f^{-1})^{-1}(K) \subset T$ is closed. But $(f^{-1})^{-1} = f$ so we must show $f(K)$ is closed. As $S$ is compact, $K \subset S$ closed is also compact. As $f$ is continuous $f(K) \subset T$ is compact and thus closed. $\blacksquare$

Test 1-3

(3a) A seq. $\{x_j\}$ is Cauchy if $\forall\, \varepsilon > 0\ \exists\, N_\varepsilon \in \mathbb{N}$ so $n, m > \tilde{N}_\varepsilon \Rightarrow d_S(x_n, x_m) < \varepsilon$.

(3b) Suppose $\{x_{n_j}\}$ is a subseq. of $\{x_n\}$, that $\{x_n\}$ is Cauchy and $\lim\limits_{j\to\infty} x_{n_j} = x_0$ exists.

$j \in \mathbb{N} \to n_j \in \mathbb{N}$ is increasing $n_j \geq j$.

Given $\varepsilon > 0$. $\exists\, J_\varepsilon$ s.t. $j > J_\varepsilon \Rightarrow d_S(x_{n_j}, x_0) < \varepsilon/2$

$\exists\, \tilde{N}_\varepsilon$ s.t $m, \ell > \tilde{N}_\varepsilon \Rightarrow d_S(x_\ell, x_m) < \varepsilon/2$.

Set $N_\varepsilon = \max(\tilde{N}_\varepsilon, J_\varepsilon)$. Then $\ell > N_\varepsilon \Rightarrow$

$$d_S(x_\ell, x_0) \leq d_S(x_\ell, x_{n_j}) + d_S(x_{n_j}, x_0)$$

Choosing $j > N_\varepsilon \Rightarrow n_j > \tilde{N}_\varepsilon$ (Cauchy cond.)

and $j > J_\varepsilon$ (conv. of subseq. cond.), we get

$$d_S(x_\ell, x_0) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

so $x_\ell \to x_0$ in $S$. ∎

Test 1-4

(4.) A cmp. metric sp. is complete

let $\{x_j\}$ be Cauchy in $(S, d_S)$. If not convergent
no pt. is a limit pt of the set $\{x_j\}$.
Every pt. $p \in S$ $\exists$ $N_{\varepsilon_p}(p)$ containing

at most finitely many pts of $\{x_j\}$

$S = \bigcup_{p \in S} N_{\varepsilon_p}(p)$ Since $S$ is compact

$S = \bigcup_{\substack{p_j \in S \\ j=1,\dots,k}} N_{\varepsilon_p}(p_j)$. Then $\{x_j\}$ has finitely

many elements & $d(x_j, x_l) > \varepsilon$ $\forall j, l$

So can't be Cauchy. ∎