MA641 Differential Geometry
Fall 2010
Problem Set 3
DUE: Friday, 29 October 2010

(1) Prove that the Lie derivative defined on pages 28-29 of do Carmo for all smooth vector fields X, Y and points $p \in M$ by

$$(L_X Y)_p =: \lim_{t\to 0} \frac{1}{t}[Y_p - (d\phi_t)_{\phi_{-t}(p)} Y_{\phi_t(p)}]$$

is given by $L_X Y = [X, Y]$. Here, the map ϕ_t is the local flow generated by X. Provide all the details of the proof.

(2) Prove that $SU(2)$ is simply connected and homeomorphic to the three sphere S^3 using the map:

$$x \in S^3 \to x_1 I_2 + i(x_2, x_3, x_4) \cdot \overline{\sigma},$$

where $\overline{\sigma}$ is the vector formed from the three Pauli matrices.

(3) Continuation of do Carmo, page 46, #4: Prove that the mappings

$$\alpha_A(z) = \frac{az+b}{cz+d}$$

for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R),$$

are isometries of the left-invariant Riemannian metric constructed in the problem.

(4) Let $\mathbb{R}/\mathbb{Z}$ denote the set of equivalence classes $\{[x] \mid x \in \mathbb{R}, y \in [x] \text{ iff } y = x + n, n \in \mathbb{Z}\}$. Show that the circle $S^1 \equiv \{(x, y) \mid x^2 + y^2 = 1\}$ is diffeomorphic to $\mathbb{R}/\mathbb{Z}$ by studying the map $f([x]) = (\cos(2\pi x), \sin(2\pi x))$. Now consider the product $S^1 \times S^1$ (see do Carmo, page 31, exercise 1). Show that $S^1 \times S^1$ is diffeomorphic to the torus T^2 defined on pages 23-24.