MA641 Differential Geometry
Fall 2010
Problem Set 6
DUE: Wednesday, 15 December 2010

(1) Suppose a two-dimensional manifold (M, g) has a metric (in global (x_1, x_2) coordinates)

$$g_{11} = g_{22} = e^{-2f}; g_{12} = 0,$$

where f is a smooth function of the coordinates.

(a) Find the Christoffel symbols Γ^k_{ij}. There are only four independent nonzero symbols: $\Gamma^1_{11}, \Gamma^1_{12}, \Gamma^1_{22}, \Gamma^2_{22}$. Note that by symmetry: $\Gamma^k_{12} = \Gamma^k_{21}$. In this case, we also have $\Gamma^k_{ij} = -\Gamma^j_{ik}$ so that $\Gamma^1_{22} = -\Gamma^2_{21} = -\Gamma^2_{12}$ and $\Gamma^1_{12} = -\Gamma^2_{11}$.

(b) Show that the only nonzero component of the curvature tensor R_{ijkl} is R_{1212} (up to all the usual symmetries). Show that in this case, we have

$$R_{1212} = e^{-2f}\Delta f, \text{ where } \Delta f = \partial^2_{x_1} f + \partial^2_{x_2} f.$$

(c) Compute the Ricci and scalar curvatures for (M, g).

(2) Apply the results of problem 1 to hyperbolic space H^2 using the upper half-space model. In this case, the metric is $g_{11} = g_{22} = x_2^{-2}$ and $g_{12} = 0$. What are the sectional curvatures?

(3) Let $\mathbb{R}/\mathbb{Z}$ denote the set of equivalence classes $\{[x] \mid x \in \mathbb{R}, y \in [x] \text{ iff } y = x + n, n \in \mathbb{Z}\}$. Show that the circle $S^1 \equiv \{(x, y) \mid x^2 + y^2 = 1\}$ is diffeomorphic to $\mathbb{R}/\mathbb{Z}$ by studying the map $f([x]) = (\cos(2\pi x), \sin(2\pi x))$. Now consider the product $S^1 \times S^1$ (see do Carmo, page 31, exercise 1). Show that $S^1 \times S^1$ is diffeomorphic to the torus T^2 defined on pages 23-24.