MA 671

PS1-1

Solutions to PS 1

1 a) $\{z \mid |z-z_1| = |z-z_2|,\ z_1, z_2 \in \mathbb{C} \text{ fixed}\}$

Translate $z \to z+z_1$ so $|z| = |z-w|$

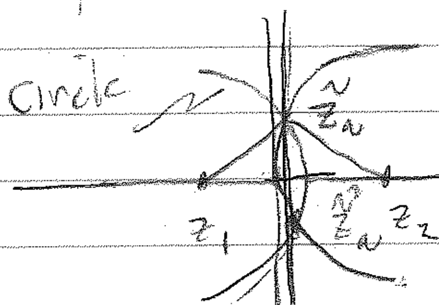
with $w = (z_2 - z_1)$

Square: $|z|^2 = |z-w|^2$ or $x^2+y^2 = (x-w_1)^2 + (y-w_2)^2$

$$0 = -2xw_1 - 2yw_2 + w_1^2 + w_2^2$$

$$2w_1 x + 2w_2 y = (w_1^2 + w_2^2)$$

a straight line. Geometrically



Rotate $\mathbb{C}$ so z_1 & z_2 lie on the Re z axis.

$|z-z_1| = a > 0$ circle

$|z-z_2| = a > 0$ circle

$\tilde{z}_a$ satisfies $|z_1 - \tilde{z}_a| = |z_2 - \tilde{z}_a|$

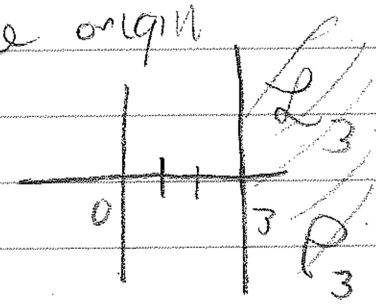
as does $\tilde{\tilde{z}}_a$

the line through $\tilde{z}_a$ & $\tilde{\tilde{z}}_a$ is $\perp$ to Re z-axis.

b.) $\frac{1}{z} = \bar{z}$ or $1 = |z|^2$ Circle of radius 1 about the origin

For ex, $z = re^{i\theta}$ then $|z|^2 = r^2 = 1$.

c.) $\{Re(z) = 3\} = \{3+iy \mid y \in \mathbb{R}\}$
$= \mathcal{L}_3$



d) $Re(z) > c$ or $\geq c$ Half plane either open or closed

ex. $c = 3$, P_3 or $\bar{P}_3$ in the figure

P51-2

2. $\langle\cdot,\cdot\rangle$ IP on $\mathbb{R}^2$ & $(\cdot,\cdot)$ IP on $\mathbb{C}$, that is

- $Z, W \in \mathbb{R}^2$ $\langle Z, W\rangle = z_1 w_1 + z_2 w_2$
- $z, w \in \mathbb{C}$ $(z, w) = \bar{z} w$ so $\overline{(z,w)} = z\bar{w} = (w, z)$

and it is linear in the 2^{nd} place

& conjugate-linear in the 1^{st} place.

Compute

$$(z, w) = (z_1 - i z_2)(w_1 + i w_2) = z_1 w_1 + z_2 w_2$$
$$+ i(z_1 w_2 - z_2 w_1)$$
$$= \mathrm{Re}(z, w) + i\,\mathrm{Im}(z, w)$$

so $\mathrm{Re}(z, w) = z_1 w_1 + z_2 w_2 = \langle Z, W\rangle$

(Note $\mathrm{Re}\,\zeta = \frac{1}{2}(\zeta + \bar{\zeta})$ any $\zeta \in \mathbb{C}$)

PS1-3

3,

(a) $z, w \in \mathbb{C}$ $\quad \bar{w}z \neq 0$ then $|z|<1$ & $|w|<1 \Rightarrow \left|\frac{w-z}{1-\bar{w}z}\right|<1$

Reduction: $w = r_w e^{i\theta_w}$ with $\theta_w \in [0, 2\pi)$, $r_w = |w| < 1$

$e^{-i\theta_w} w = r_w$ real. note that

$$\left|\frac{e^{-i\theta_w}(w-z)}{1-(e^{+i\theta_w}\bar{w})(e^{-i\theta_w}z)}\right| = \left|\frac{w-z}{1-\bar{w}z}\right|$$

$$= \left|\frac{r_w - \tilde{z}}{1 - r_w \tilde{z}}\right|$$

where $\tilde{z} = e^{-i\theta_w} z \in \mathbb{C}$. So it suffices to consider $w = r_w \in \mathbb{R}^+$ with $r_w > 0$.

Consequently

(a) (i) $\left|\frac{r-z}{1-rz}\right| < 1 \iff |r-z| < |1-rz|$

$\iff (r-x)^2 + y^2 < (1-rx)^2 + (ry)^2$

Now $|z| = \sqrt{x^2+y^2} < 1$

and $r^2 - 2rx + x^2 + y^2 < 1 - 2rx + r^2x^2 + r^2y^2$

$r^2 + (x^2+y^2) < 1 + r^2(x^2+y^2)$

$(1-r^2)|z|^2 < 1 - r^2$

$|z|^2 < 1$

Whence $\left|\frac{r-z}{1-rz}\right| < 1$.

2

(ii) $|z| = 1$ or $|w| = 1$ $\quad \left|\frac{w-z}{1-\bar{w}z}\right| = 1$

Suppose $|w| = r_w = 1$ then $\left|\frac{1-z}{1-z}\right| = 1$

Suppose $|z| = 1$ then the calc in (i) holds with equality replacing $\leq$

PS1-4

(b) $F(z) = \frac{w-z}{1-\bar{w}z}$ maps $\mathbb{D}$ into $\mathbb{D}$ $(|w|<1)$

(i) Fix $w \in \overline{\mathbb{D}}$. If $|w| < 1$ then $F: \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ by part (a). F is analytic: It is a rational fnc of z and only possible singularity is $1-\bar{w}z = 0$ so $z = \frac{1}{\bar{w}}$. Since $|w| < 1$

$|z| = \frac{1}{|w|} > 1$ so the singularity is outside

of $\mathbb{D}$. So $F: \mathbb{D} \to \mathbb{D}$ is analytic

(ii) $F(0) = w$ & $F(w) = 0$

(iii) $|F(z)| = 1$ if $|z| = 1$ proved above

(iv) F is bijective.

It is easy to compute $F \circ F = \mathrm{Id}$, that is:

$$F \circ F(z) = \frac{w - F(z)}{1-\bar{w}F(z)} = \frac{w - \left(\frac{w-z}{1-\bar{w}z}\right)}{1-\left(\frac{|w|^2-\bar{w}z}{1-\bar{w}z}\right)}$$

$$= \frac{w-|w|^2 z - w + z}{1-\bar{w}z-|w|^2+\bar{w}z} = \frac{z(1-|w|^2)}{1-|w|^2} = z.$$

So $F^{-1} = F : \mathbb{D} \to \mathbb{D}$.

NOTE For (a): $|z-w|^2 < |1-\bar{w}z|^2$

$|z|^2 + |w|^2 < 1 + |z|^2|w|^2$

$0 \leq 1 - |w|^2 - |z|^2 + |z|^2|w|^2 = (1-|z|^2)(1-|w|^2)$

PS1-5

$f: \Omega \to \mathbb{C}$ analytic

If any one of the following hold then $f = \text{cnst}$ on Ω

(a) $\text{Re} f = u = \text{cnst}$. C-R $\partial_x u = \partial_y v = 0$

$\partial_y u = -\partial_x v = 0$

$\Rightarrow \partial_y v = 0 = \partial_x v$.

Solve these : $v(x,y) - v(x,y_0) = \int_{y_0}^{y} \partial_{\tilde{y}} v(x,\tilde{y})\, d\tilde{y} = 0$

$\Rightarrow$ v is independent of y in Ω so

$v(x,y) = h(x)$.

Since $\partial_x v = h'(x) = 0 \Rightarrow h(x) = \text{cnst}$.

Hence $v(x,y) = \text{cnst}$. Since $u(x,y) = \text{cnst}$.

$\Rightarrow f = u + iv = \text{cnst}$. in Ω.

(b) $\text{Im}(f) = v = \text{cnst}$. Same argument as in (a)

(c) $|f|^2 = \text{cnst}$ so $u^2 + v^2 = \text{cnst}$.

$\partial_x |f|^2 = 2(u u_x + v v_x) = 0$

$\partial_y |f|^2 = 2(u u_y + v v_y) = 0$

2 ways to proceed:

(1) $\partial_x (u u_x + v v_x) = u_x^2 + u u_{xx} + v_x^2 + v v_{xx} = 0$

$\partial_y (u u_y + v v_y) = u_y^2 + u u_{yy} + v_y^2 + v v_{yy} = 0$

Add:

$[u_x^2 + u_y^2] + u[u_{xx} + u_{yy}] + [v_x^2 + v_y^2]$

$+ v[v_{xx} + v_{yy}] = 0$

u & v are <u>harmonic</u> so $u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$

PS1-6

So $u_x^2 = 0 = u_y^2$ & $v_x^2 = 0 = v_y^2$

Proceed as above $\Rightarrow$ $u = \text{cnst}$, $v = \text{cnst}$.

(2) look at ratios: $\frac{u}{v} = -\frac{v_x}{u_x}$ & $\frac{u}{v} = -\frac{v_y}{u_y}$

$(R \Rightarrow \frac{u_y}{u_x} = \frac{-u_x}{u_y} \Rightarrow u_y^2 + u_x^2 = 0$

so $u_x = 0 = u_y$

and similarly for v.