MA671

Solutions to PS 2

10. $\Delta = \partial_x^2 + \partial_y^2 \qquad z = x+iy \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Compute $4\,\partial_z \partial_{\bar z} = (\partial_x - i\partial_y)(\partial_x + i\partial_y) = \partial_x^2 + \partial_y^2 = \Delta$

Similarly $4\,\partial_{\bar z}\partial_z = 4(\partial_z \partial_{\bar z})^* = \Delta$ or compute as above

11. $f = u+iv$ analytic C-R $\quad \partial_x u = \partial_y v \Rightarrow \partial_x^2 u = \partial_{xy}^2 v$

$$\partial_y u = -\partial_x v \Rightarrow \partial_y^2 u = -\partial_{xy}^2 v$$

add these : $\partial_x^2 u + \partial_y^2 u = 0$.

Similar calc. for v.

$4\partial_z \partial_{\bar z} f = \Delta u + i\Delta v = 0$ also.

12 $(1-z)^{-1} = \sum_{j=0}^{\infty} z^j$ Conv. unif & abs in $\mathbb{D}$.

The PS can be differentiated term-by-term.

$$\frac{d}{dz}(1-z)^{-1} = \frac{1}{(1-z)^2} = \sum_{j=0}^{\infty} j z^{j-1}$$

By induction; suppose

$$(1-z)^{-(m-1)} = \frac{1}{(m-1)!} D_z^{m-1}(1-z)^{-1} = \sum_{j=0}^{\infty} a_j(m-1)\, z^j$$

where $a_j(m-1) \sim \frac{1}{(m-2)!} j^{m-2}$

then

$$(1-z)^{-m} = \frac{1}{m-1} D_z (1-z)^{-(m-1)} = \sum_{j=0}^{\infty} \frac{a_j(m-1)}{m-1}\, j\, z^{j-1}$$

$$= \sum_{k=-1}^{\infty} \frac{a_{k+1}(m-1)}{m-1}(k+1)\, z^k$$

PS2-2

$$a_k(m) = a_{k+1}(m-1)(k+1)$$
$$\sim \frac{1}{(m-2)!\,(m-1)}(k+1)(k+1)^{m-2}$$

Consequently
$$a_k(m) \sim \frac{1}{(m-1)!} k^{m-1}$$

23. $f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x^2} & x > 0 \end{cases}$

note: ① $\lim_{x \to 0} f(x) = 0$.

② $f \in C^\infty(\mathbb{R})$, $f^{(j)}(0) = 0$. (Clear for $x \to 0^-$)

Why? for $x > 0$,
$$f(x) = e^{-h(x)} = g(h(x)),\ g(x) = e^{-x^2},\ h(x) = \frac{1}{x}.$$
and each is C^∞ for $x > 0$.

To check the behavior at 0, $f^{(m)}(x)$ contains terms like $\frac{1}{x^p} e^{-1/x^2}$. Write $u = \frac{1}{x}$ so $x \to 0^+$ corresponds to $u \to \infty$. $\lim_{u \to \infty} u^p e^{-u^2} = \lim_{u \to \infty} \frac{u^p}{e^{u^2}} = \frac{\infty}{\infty}$

Take p-derivatives to get
$$\lim_{u \to \infty} \frac{u^p}{e^{u^2}} = \lim_{u \to \infty} \left(\frac{1}{u^2 e^{u^2}}\right)$$ up to a constant

this has limit zero.

Whence, if f had a Taylor exp at zero, $f(x) = \sum_{j=0}^{\infty} x^j a_j$ where $a_j = \frac{f^{(j)}(0)}{j!} = 0$. But f is not identically 0.

so $f \in C^\infty(\mathbb{R})$ is not the restriction of an analytic function on some $B_r(0)$, $r>0$, to $\mathbb{R}$.

24. γ smooth curve parametrized by $z(t) : [a,b] \to \mathbb{C}$ and γ^- is the reverse orientation with param. $z^-(t) = z(a+b-t)$

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

and

$$\int_{\gamma^-} f(z)\,dz = \int_a^b f(z^-(t))\,(z^-)'(t)\,dt$$

• $(z^-)'(t) = -z'(a+b-t)$

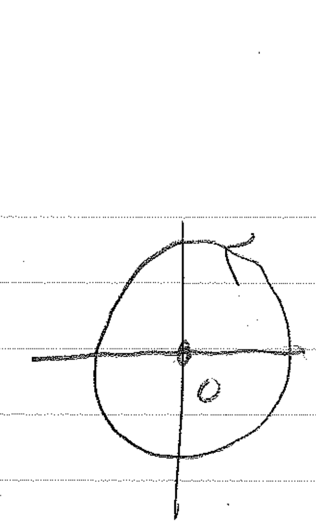
$$= -\int_a^b \cancel{f(z(a+b-t))}\, z'(a+b-t)\,dt$$

• let $s = a+b-t$ so $ds = -dt$

$$= -\int_b^a f(z(s))\,z'(s)\,(-ds) = -\int_a^b f(z(s))\,z'(s)\,ds$$

$$= -\int_\gamma f(z)\,dz.$$

Q5 a) $\displaystyle\oint_{|z|=r} z^n \, dz \qquad z = re^{i\theta}$



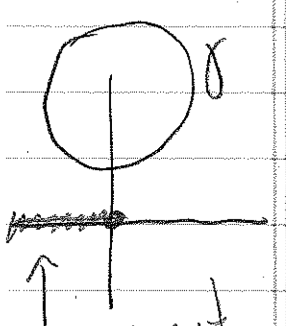
$$= \int_0^{2\pi} r^n e^{in\theta} \, i e^{i\theta} r \, d\theta$$

$$= i r^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \, d\theta$$

$$= i r^{n+1} \begin{cases} \left. \dfrac{e^{i(n+1)\theta}}{i(n+1)} \right|_0^{2\pi} = 0 & n \neq -1 \\ 2\pi & n = -1 \end{cases}$$

$$= \begin{cases} 0 & n \neq -1 \\ 2\pi i r^{n+1} & n = 0 \end{cases}$$

b) Any circle NOT enclosing 0. If $n \neq -1$, $f_n(z) = z^n$



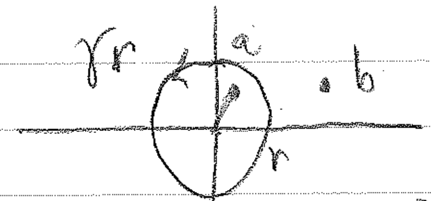
Branch cut for log

is analytic on & inside the circle γ: So $F_n(z) = \frac{1}{n+1} z^{n+1}$ is a primitive and $\displaystyle\int_\gamma f_n(z)\, dz = 0$ by the Fundamental Th. (for any closed contour in fact not containing zero). For $n = -1$, choose any branch of the log not intersecting Int γ so $F_{-1}(z) = \log z$ is analytic and an anti-derivative.

March 6th F.

PS 2-5

c.) $|a| < r < |b|$



Partial fraction : $\dfrac{1}{(z-a)(z-b)} = \left(\dfrac{1}{z-a} - \dfrac{1}{z-b}\right)\left(\dfrac{1}{a-b}\right)$

$$\oint_{\gamma_r} \frac{dz}{(z-a)(z-b)} = \frac{1}{a-b}\left\{\oint_{\gamma_r}\frac{dz}{z-a} - \oint_{\gamma_r}\frac{dz}{z-b}\right\}$$

By argument (b), $\oint_{\gamma_r} \frac{dz}{z-b} = 0$

For the 1st integral:

$$\oint_{\gamma_r} \frac{dz}{z-a} dz$$

2 possible ways to do it:

(1) Power series $z(t) = re^{it}$ so as $|a|/r < 1$

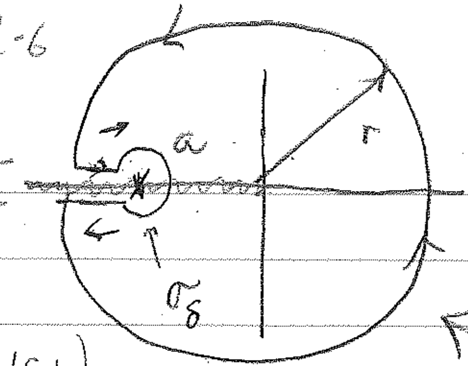
$$\oint_{\gamma} \frac{dz}{z-a} = \int_0^{2\pi} \frac{rie^{it}\,dt}{re^{it}-a} = i\int_0^{2\pi} \frac{dt}{1-\frac{a}{r}e^{it}}$$

$$= i\sum_{j=0}^{\infty}\int_0^{2\pi}\left(\frac{a}{r}\right)^j e^{-itj}\,dt$$

$$= 2\pi i$$

(2) Change of contour. Choose $\log z$ with PB (so $\arg z \in [-\pi, \pi]$)

PS 2-6



You can rotate everything so a is on the neg real axis.

Then $F(z) = \log_{PB}(z + |a|)$

is analytic on & inside the key-hole contour γ_ε

$$0 = \oint_{\gamma_\varepsilon} \frac{dz}{z+|a|} = \int_{\sigma_{\delta,\varepsilon}} \frac{dz}{z+|a|} + \int_{(\rightleftarrows)_\varepsilon} \frac{dz}{z+|a|} + \int_{\gamma_\varepsilon} \frac{dz}{z+|a|}$$

Claim: $\left(\int_{\rightarrow_\varepsilon} + \int_{\leftarrow_\varepsilon} \right) \frac{dz}{z+|a|} = 0$, so taking

$\varepsilon \to 0$: $\displaystyle\int_{\gamma_r} \frac{dz}{z-a} = \oint_{\sigma_\delta} \frac{dz}{z-a} = 2\pi i$
by part (a)

Proof of the claim:

$$\int_{\rightarrow} \frac{dz}{z-a} = \int_{-r+i\varepsilon}^{a-\delta+i\varepsilon} \frac{dz}{z-a} = \log_{PB}(a-\delta+i\varepsilon) - \log_{PB}(-r+i\varepsilon)$$

$$= \left(\log|a-\delta| + i(\pi-\varepsilon)\right) - \left(\log|r| + i(\pi-\varepsilon)\right)$$
$$= \log\left(\frac{|a-\delta|}{r}\right)$$

Similarly $$\int_{\leftarrow} \frac{dz}{z-a} = \int_{a-\delta-i\varepsilon}^{-r-i\varepsilon} \frac{dz}{z-a} = \log_{PB}(-r-i\varepsilon) - \log_{PB}(a-\delta-i\varepsilon)$$

$$= \log\left(\frac{r}{|a-\delta|}\right) + i(\varepsilon-\pi) - i(\varepsilon-\pi) = \log\frac{r}{|a-\delta|}$$
and these cancel.