MA 671

PS3-1

Solutions to Problem Set 3

1. a) If $f = u + iv$ is analytic, $\partial_x u = \partial_y v$ & $\partial_y u = -\partial_x v$

Taking

$$\partial_x^2 u = \partial_x \partial_y v = -\partial_y^2 u \Rightarrow \Delta u = 0$$

similarly for v. So u & v are harmonic on the same domain where f is analytic

b) Given $u \in C^2$ on Ω and harmonic $\Delta u = 0$ on Ω, find v harmonic so $\partial_x u = \partial_y v$ & $\partial_y u = -\partial_x v$.

Solve $\partial_y v = \partial_x u$ by integrating from $y_0 \to y$

$$(*)\ V(x,y) = \int_{y_0}^{y} \partial_x u(x,t)\,dt + g(x)$$

note: $V(x, y_0) = g(x)$

Similarly $\partial_x V = -\partial_y u$ so integrating from $x_0 \to x$

$$(*)\ V(x,y) = -\int_{x_0}^{x} \partial_y u(s,y)\,ds + h(y)$$

note: $V(x_0, y) = h(y)$

By equating the 2 results we find

$$V(x_0, y_0) = g(x_0) = h(y_0)$$

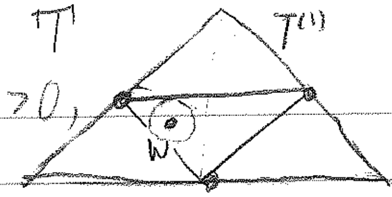
But as x_0 & y_0 vary so $z_0 = x_0 + iy_0 \in \Omega$

$g(x_0) = h(y_0)$ requires $g(x_0) = h(y_0) = $ const.

So $(*)$ defines $v(x,y)$ up to a constant.

2. pg 65 #6. $f: A \to \mathbb{C}$ analytic on $A \setminus \{w\}$, let T be any triangle with $\operatorname{Int} T \subset A$ and $w \in \operatorname{Int} T$. Suppose f

(p53-2)



$\int_{T^{(1)}} f = 0$ by Goursat

is bound on $B_r(w)$, some $r > 0$,
so $\|f(z)\| < M \ \forall z \in B_r(w)$

CA. Reduce T to $\tilde{T}_\varepsilon$, where $\tilde{T}_\varepsilon$ is a triangle, $w \in \operatorname{Int} \tilde{T}_\varepsilon$, and $|\tilde{T}_\varepsilon| < \varepsilon$, for any $\varepsilon > 0$, by a bisection procedure as in the proof of Goursat's theorem. Then, because $\int_{T'} f = 0$ for any $T' \subset \overline{\operatorname{Int} T}$ not containing w,

we get $\left|\int_T f\right| = \left|\int_{\tilde{T}_\varepsilon} f\right| \le M\varepsilon \quad \forall \varepsilon > 0$.

3. pg 65 # 7. $f : \mathbb{D} \to \mathbb{C}$. $d \overset{df}{=} \sup_{z,w \in C_r} |f(z) - f(w)|$

let $0 < r < 1$ & C_r circle of rad. r

By Cauchy's integral formula $f'(0) = \frac{1}{2\pi i} \oint_{C_r} \frac{f(w)}{w^2} dw$

C_r $w = re^{it}$ & $-w = re^{i(t+\pi)}$ so $-\oint_{C_r} \frac{f(-w)\,dw}{w^2}$

$$= -\oint_0^{2\pi} \frac{f(re^{i(t+\pi)})}{r^2 e^{2it}} \cdot ire^{it}\, dt$$

$u = t + \pi$

$$= -\int_\pi^{3\pi} \frac{f(re^{iu})}{r^2 e^{2iu} e^{-2\pi i}}\, ri e^{i(u-\pi)}\, du$$

$$= -\int_\pi^{3\pi} \frac{f(re^{iu})}{r^2 e^{2iu}} (-1)\, ri e^{iu}\, du = \oint_{C_r} \frac{f(w)}{w^2}\, dw$$

$$2f'(0) = \frac{1}{2\pi i}\oint_{C_r} \frac{f(w)-f(-w)}{w^2}\,dw$$

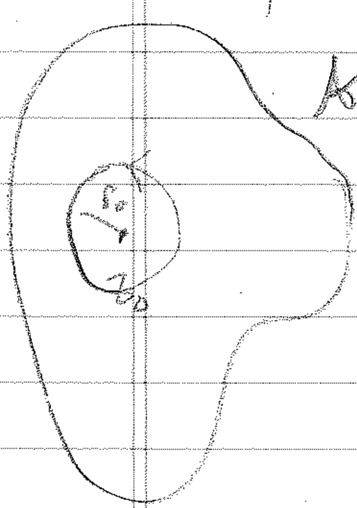
and

$$2|f'(0)| \le \frac{1}{2\pi}\,\frac{d}{r^2}\,2\pi r = \frac{d}{r}.$$

since $0<r<1$ we have

$$2|f'(0)| \le d.$$

4. u harmonic in a region containing $\overline{\mathbb{D}_{r_0}(z_0)}$. There is a harmonic conjugate v so $f = u + iv$ is analytic in a neighborhood of $\overline{\mathbb{D}_{r_0}(z_0)}$. By the Cauchy integral formula:



$$f(z_0) = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}_{r_0}(z_0)} \frac{f(w)}{w-z_0}, \quad w = z_0 + r_0 e^{i\theta}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta$$

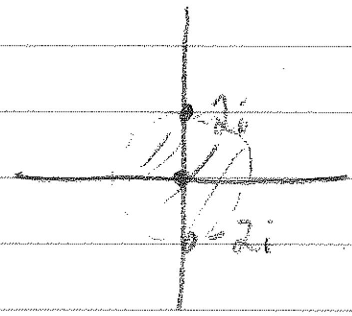
$$\boxed{f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r_0 e^{i\theta})\,d\theta}$$

Mean value property for analytic fncs.
Take the real part:

$$f(z_0) = u(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + r_0\cos\theta,\, y_0 + r_0\sin\theta)\,d\theta$$

(PS3-4)

5. $f(z) = (z^2+4)^{-1}$ 2 singularities $z = \pm 2i$



$R = 2$ because any large R would mean these bad pts are inside the disk.

$$f(z) = \frac{4}{(z/2)^2+1}$$ note that $|z/2| < 1$ if $|z| < 2$

$$= 4 \sum_{j=0}^{\infty} (-1)^j \left(\frac{z}{2}\right)^{2j}$$ by the geometric series formula.

Check the radius of convergence:

$$f(z) = 4 \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j}} z^{2j}$$

Ratio: $$\left| \frac{\frac{(-1)^{j+1}}{2^{2(j+1)}} z^{2(j+1)}}{\frac{(-1)^j}{2^{2j}} z^{2j}} \right| = \frac{|z|^2}{2^2} < 1$$

$\Rightarrow |z| < 2 \Rightarrow R = 2.$