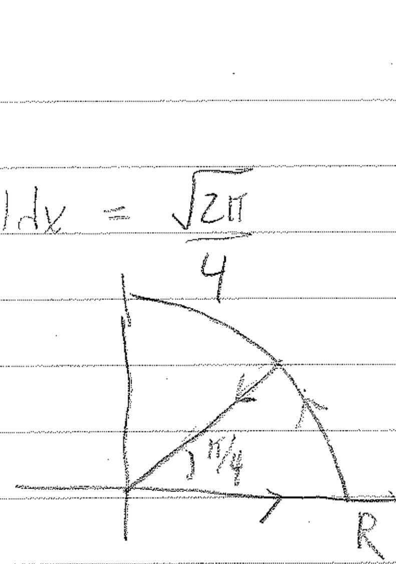
MA 671

PS4-1

Solutions to PS 4

1. Show: $\displaystyle\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}$



$$\oint_{\gamma_R} e^{-z^2}\,dz = 0$$

since e^{-z^2} is entire.

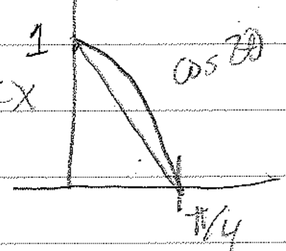
$$(*)\quad 0 = \int_0^R e^{-x^2}dx + \underbrace{\int_0^{\pi/4} e^{-R^2e^{2i\theta}}\, iRe^{i\theta}\,d\theta}_{z = Re^{i\theta}} + \underbrace{\int_R^0 e^{-ir^2} e^{i\frac{\pi}{4}}dr}_{z = e^{i\pi/4}r}$$

Note the role of $\pi/4$: $\operatorname{Re} e^{ir^2} = \cos r^2$ & $\operatorname{Im} e^{ir^2} = \sin r^2$

(i) $\displaystyle I_R \equiv \left|\int_0^{\pi/4} e^{-R^2e^{2i\theta}}\, iRe^{i\theta}d\theta\right| \le R\int_0^{\pi/4} e^{-R^2\cos 2\theta}\,d\theta$

on $[0, \pi/4]$, $\cos 2\theta$ ranges from $+1$ to 0. The problem is near $\theta = \frac{\pi}{4}$.

But $\cos 2\theta$ is convex so we bound it from below by the secant.



$$\cos 2\theta \ge \left(1 - \frac{4}{\pi}\theta\right) \ge 0.$$

$$e^{-R^2\cos 2\theta} \le e^{-R^2} e^{\frac{4}{\pi}R^2\theta}$$

$$\int_0^{\pi/4} e^{\frac{4}{\pi}R^2\theta}\,d\theta = \left(e^{R^2}-1\right)\frac{\pi}{4R^2}$$

so $\displaystyle I_R \le \frac{\pi}{4R} e^{-R^2}\left(e^{R^2}-1\right) = \frac{\pi}{4R}\left(1 - e^{-R^2}\right) \to 0$ as $R \to \infty$

(ii) Rearrange $(*)$ $\displaystyle\int_0^\infty e^{-x^2}dx = e^{i\frac{\pi}{4}}\int_0^R e^{ix^2}dx$

PS4-2

Problem 2

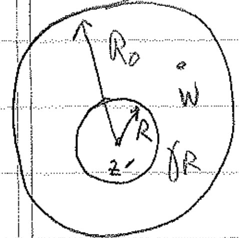
(S&S Problem 11) a) $0 < R < R_0 \,,\ |z| < R$

Show $(*)$ $f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\,\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi$

Cauchy Integral formula. If $z \in \mathrm{Int}\, C_R$; "$C_R = \{Re^{i\varphi} \mid \varphi \in [0,2\pi]\}$"

$$f(z) = \frac{1}{2\pi i}\int^{2\pi} \frac{f(Re^{i\varphi})}{Re^{i\varphi}-z}\, Rie^{i\varphi}d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\left(\frac{Re^{i\varphi}}{Re^{i\varphi}-z}\right)d\varphi$$



$|z| < R$. Define $w = \frac{R^2}{\bar{z}}$ so $|w| = \frac{R^2}{|z|} > R$

so $\displaystyle\int_{\gamma_R} \frac{f(\zeta)}{\zeta - w}\, d\zeta = 0$

After some manipulation: $0 = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\left(\frac{\bar{z}}{\bar{z}-Re^{-i\varphi}}\right)d\varphi$

Set $\zeta = Re^{i\varphi}$ so taking the difference,

$$\frac{Re^{i\varphi}}{Re^{i\varphi}-z} - \frac{\bar{z}}{\bar{z}-Re^{-i\varphi}} = \frac{|z|^2-|\zeta|^2}{|z-\zeta|^2}$$

Finally note: $\frac{1}{2}\left(\frac{\zeta+z}{\zeta-z}+\frac{\bar{\zeta}+\bar{z}}{\bar{\zeta}-\bar{z}}\right) = \frac{|z|^2-|\zeta|^2}{|z-\zeta|^2}$

$$= \mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)$$

We get: $f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\left\{\frac{Re^{i\varphi}}{Re^{i\varphi}-z} + \frac{\bar{z}}{\bar{z}-Re^{-i\varphi}}\right\}d\varphi$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\,\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi$$

(PS4-3)

b) Let $z = re^{i\theta}$, with, $0 < |z| = r < R$. Then,

(**) $\operatorname{Re}\left(\dfrac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right) = \dfrac{r^2 - R^2}{|z - Re^{i\varphi}|^2}$ from the calculation in part (a).

Compute $|z - Re^{i\varphi}|^2 = (\bar{z} - Re^{-i\varphi})(z - Re^{i\varphi})$

$= |z|^2 - Rze^{-i\varphi} - \bar{z}Re^{i\varphi} + R^2$

$= r^2 - Rr[e^{i(\theta-\varphi)} + e^{-i(\theta-\varphi)}] + R^2$

$= R^2 - 2rR\cos\gamma + r^2$

where $\gamma = \theta - \varphi$.

so (**) is $(1 - R^2)(R^2 - 2rR\cos\gamma + r^2)^{-1}$.

Problem (12b) Show if $\Delta u = 0$ on $\mathbb{D}_1(0)$ & $u|_{\partial\mathbb{D}_1(0)}$ is continuous then

$$u(z) = u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)\, u(e^{i\varphi})\, d\varphi$$

where the Poisson kernel is $P_r(\gamma) = \dfrac{1 - r^2}{1 - 2r\cos\gamma + r^2}$

Proof u harmonic means there is a harmonic conjugate v so $u + iv$ is analytic on $\mathbb{D}_1(0)$. Formula (*) in (a) applies for $0 < r < R < 1$. We get:

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\varphi})\operatorname{Re}\left(\frac{Re^{i\varphi}+re^{i\theta}}{Re^{i\varphi}-re^{i\theta}}\right) d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\varphi) + r^2}\right) u(Re^{i\varphi})\, d\varphi$$

Since u is continuous on $\overline{\mathbb{D}_1(0)}$, take $R \to 1$ and obtain

$$= \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)\, u(e^{i\varphi})\, d\varphi.$$

(PS4-4)

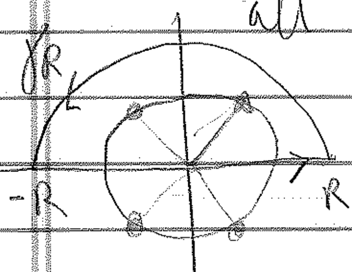
3. (S&S p103 #2) Compute $\int_{\mathbb{R}} \frac{dx}{1+x^4}$

$f(z) = (1+z^4)^{-1}$ has 4 poles at $z^4 = -1 = e^{i\pi}$

$z_k = e^{i(\pi/4 + \frac{2\pi}{4}k)}$, $k = 0,1,2,3$.

$z_0 = e^{i\pi/4}$, $z_1 = e^{3\pi/4 i}$, $z_2 = e^{i5\pi/4}$, $z_3 = e^{i7\pi/4}$

all satisfy $|z_k| = 1$.



$|f(z)| \lesssim \frac{1}{|z|^4}$ $\quad |z| \to \infty$

Use contour $\gamma_R = [-R,R] \cup \{Re^{i\theta}, \theta \in [0,\pi]\}$
$= [-R,R] \cup \gamma_R^+$

(Can also use γ_R^-)

$R >> 1$: $\oint_{\gamma_R} f(z)\,dz = 2\pi i \{ \mathrm{Res}(f, e^{i\pi/4}) + \mathrm{Res}(f, e^{3i\pi/4}) \}$

Residues: Simplest $f(z) = \frac{h(z)}{g(z)}$; $\begin{cases} h(z) = 1 \\ g'(z) = 4z^3 \quad g'(z_k) \neq 0. \end{cases}$

Simple poles at each z_k

$\mathrm{Res}(f, e^{i\pi/4}) = \frac{1}{4z_0^3} = \frac{1}{4} e^{-3\pi/4 i}$

$\mathrm{Res}(f, e^{3i\pi/4}) = \frac{1}{4} e^{-9\pi i/4} = \frac{1}{4} e^{-i\pi/4}$

Calc. $\oint_{\gamma_R} f = \int_{-R}^{R} f(x)\,dx + \int_0^{\pi} \frac{1}{1+R^4 e^{4i\theta}} iRe^{i\theta}\,d\theta$

Semi-circle: $\int_0^{\pi} \frac{R\,d\theta}{|1+R^4 e^{4i\theta}|} < \frac{C}{R^3} \to 0$

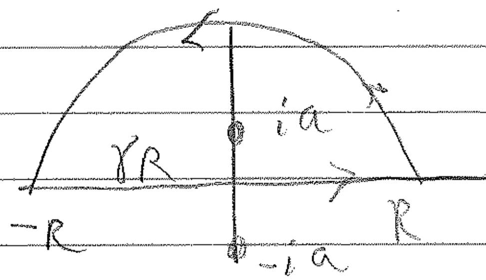
$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi i}{2} \left(\frac{\sqrt{2}}{2}(1-i) + \frac{\sqrt{2}}{2}(-1-i) \right) = \frac{\sqrt{2}\pi}{2}.$

PS4-5

4. (S&S pg 103 #3) $\displaystyle\int_{\mathbb{R}} \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{a},\ a>0$

Let $f(z) = \dfrac{e^{zi}}{z^2+a^2}$ so $|f(z)| \le \dfrac{e^{-y}}{|z^2+a^2|} \to 0$ in UHP

and $\operatorname{Re} f(x) = \dfrac{\cos x}{x^2+a^2}$.



Close the contour in UHP

$z^2+a^2=0$

$z=\pm ia,\ a>0$

$z^2+a^2 = (z-ia)(z+ia)$ 2 simple poles

$$\int_{\gamma_R} f = 2\pi i \operatorname{Res}(f, ia) = 2\pi i \frac{e^{-a}}{2ia} = \frac{\pi e^{-a}}{a}$$

and $$\left|\int_{\gamma_R^+} f\right| = \left|\int_0^\pi \frac{R e^{-(\cos\theta)R}}{|R^2 e^{2i\theta}+a^2|}\,d\theta\right|$$

$$\le \frac{c}{R} \to 0$$

$$\Rightarrow \int_{\mathbb{R}} \frac{e^{ix}}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{a} = \int_{\mathbb{R}} \frac{\cos x}{x^2+a^2}\,dx.$$