MA 671 Spring 2020
Hour Test - 100 Points
26 February 2020

INSTRUCTIONS: PLEASE WORK ALL THE PROBLEMS BELOW. EACH PROBLEM IS WORTH 20 POINTS. NO BOOKS, PAPERS, OR NOTES ARE ALLOWED.

NAME: Solutions

**Problem1.** Compute the following integrals. Clearly state your reasoning.

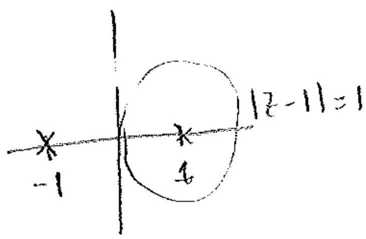
i.
$$\int_{|z-1|=1} \frac{e^z}{(z^2-1)^2}\,dz$$

ii.
$$\int_\gamma (x^2+iy^2)\,dz$$

where $\gamma$ is a straight line from 0 to $1+i$.

(i) Cauchy's formula with $k=1$

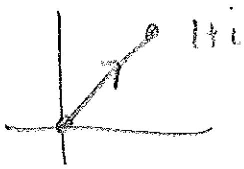
$$\int_{|z-1|=1} \frac{e^z (z+1)^{-2}}{(z-1)^2}\,dz = 2\pi i \frac{d}{dz}\left(\frac{e^z}{(z+1)^2}\right)\Bigg|_{z=1}$$




$$= 2\pi i\left(\frac{e^z}{(z+1)^2} - \frac{2e^z}{(z+1)^3}\right)_{z=1}$$

$$= 2\pi e i\left(\frac{1}{4}-\frac{1}{4}\right) = 0$$

(ii)




parameterise by $x \in [0,1]$ then $y=x$

$$\int_\gamma (x^2+iy^2)(dx+idy) = \int_0^1 (x^2+iy^2)\,dx + i\int_0^1 (x^2+iy^2)\,dy$$

$$= \frac{1}{3}(1+i) + \frac{i}{3}(1+i) = \frac{1}{3}(1+i)^2$$

or $z(t)=t(1+i)$, $t\in[0,1]$, so $x(t)=t=y(t)$, $z'(t)=(1+i)$

$$\Rightarrow \int_\gamma (x^2(t)+iy(t)^2)\,z'(t)\,dt = \int_0^1 (1+i)(t^2+it^2)$$

$$= \frac{1}{3}(1+i)^2 = \frac{2}{3}i$$

Note that since $x^2+iy^2$ isn't analytic the integral isn't path independent

**Problem 2.**

i. Let $f_j : \mathcal{A} \to C$ be a sequence of functions on $\mathcal{A}$ that converge uniformly $\mathcal{A}$ to $f$. Let $\gamma \subset \mathcal{A}$ be a smooth curve in $\mathcal{A}$ so that $|\gamma| < \infty$. Then we have

$$\lim_{j\to\infty} \int_\gamma f_j = \int_\gamma f.$$

ii. Now suppose that each $f_j$ is analytic and $\mathcal{A} = D_r(z_0)$, a disk of radius $r > 0$ about $z_0 \in C$, and that $f_j \to f$ uniformly on $D_r(z_0)$. Prove that the limit function $f$ is analytic on $D_r(z_0)$. HINT: Think about Goursat's Theorem for $f_j$, apply part (i), and think about what Goursat's Theorem implies for anti-derivatives.

(i) Estimate:
$$\left|\int_\gamma f_j - \int_\gamma f\right| \le \int_\gamma |f_j(z) - f(z)|\,|dz| \le \sup_{z\in\gamma} |f_j(z) - f(z)| \cdot |\gamma| \le \left(\sup_{z\in A} |f_j(z) - f(z)|\right)|\gamma|$$

Given $\varepsilon > 0$ uniform conv. means $\exists J_\varepsilon > 0$ s.t. $j > J_\varepsilon \Rightarrow \|f_j - f\|_{\infty, A} = \sup_{z\in A} |f_j(z) - f(z)| < \varepsilon$

so $\left|\int_\gamma f_j - \int_\gamma f\right| \le \varepsilon|\gamma|$ $j > J_\varepsilon$, $|\gamma| < \infty$.

This implies the convergenc of the integrals.

(ii) For each $T$ with $\overline{\mathrm{Int}\,T} \subset D_r(z_0)$, a triangle, since $f_j$ is analytic, Goursat's Theorem implies $\oint_T f_j = 0$. By part (i), we get $\forall\, T$ triangles with $\mathrm{Int}(T) \subset D_r(z_0)$, $\oint_T f = 0$. We used this condition to prove that for $z_1, z \in D_r(z_0)$, $F(z_2) \equiv \int_{z_1}^{z} f(s)\,ds$ is independent of the paths consisting of finitely many horizontal & vertical segments in $A$ connecting $z_1$ to $z$. We also proved $F$ is analytic on $A$ and $F'(z) = f(z)$. Since $F'$ is analytic so is $f$.

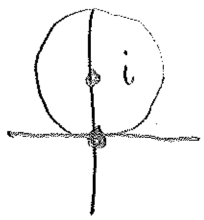
**Problem 3.**

Find the Taylor Series of the following functions and the corresponding radii of convergence.

i. $f(z)$ about $z = i$ where
$$f(z) = \frac{1}{z^2}.$$

ii. $g(z)$ about $z = 0$ where
$$g(z) = \frac{z}{z^2 - 1}.$$

(i)




$f$ is singular at $0$ so $R = 1$

$\frac{1}{z^2} = -\frac{d}{dz}\left(\frac{1}{z}\right)$. Study $\frac{1}{z} = \frac{1}{i + (z-i)} = \frac{-i}{1 - i(z-i)}$

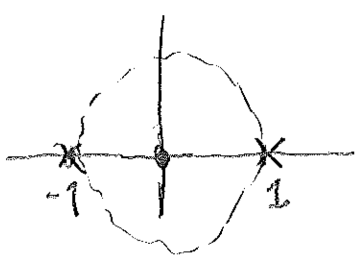
Since $|i(z-i)| < 1$, the geometric series gives $\frac{1}{z} = (-i)\sum_{k=0}^{\infty} i^k (z-i)^k$

Note: You can also compute the coefficients: $\frac{1}{z^2} = \sum_{k=0}^{\infty} \frac{f^{(k)}(i)}{k!}(z-i)^k$,

so $\frac{1}{z^2} = i\sum_{k=1}^{\infty} k i^k (z-i)^{k-1} = \sum_{m=0}^{\infty} (m+1) i^{m+2} (z-i)^m$

Check: $\lim_{m\to\infty} \left|\frac{(m+2)\, i^{m+3}}{(m+1)\, i^{m+2}}\right| = 1$ so $R = 1$.

(ii) $g(z) = \frac{z}{(z-1)(z+1)}$. $g$ is singular at $\pm 1$ so $R = 1$




$g(z) = \frac{z}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right) = \frac{z}{2}\left(-\sum_{m=0}^{\infty} z^m - \sum_{m=0}^{\infty} (-1)^m z^m\right)$

since $|z| < 1$.

If $m$ is odd, the terms cancel so set $m = 2j$!

$g(z) = -\frac{z}{2}\sum_{j=0}^{\infty} (2 \cdot z^{2j}) = -\sum_{j=0}^{\infty} z^{2j+1}$

(note: $g(x) < 0$ for $-1 < x < 1$ )



One sees $R = 1$.

Note that you can also use: $\frac{-1}{1-z^2} = -\sum_{j=0}^{\infty} z^{2j}$

for $|z| < 1$ so

$$g(z) = -\sum_{j=0}^{\infty} z^{2j+1}$$

**Problem 4.** Suppose a function $f(z)$ is entire and satisfies the bound $|f(z)| \geq 1$ for all $z \in C$. Prove that $f(z)$ is a non-zero constant.

Proof $|f(z)| \geq 1$ means $f(z)$ is bounded away from zero, so $\frac{1}{f(z)}$ is defined and $\left|\frac{1}{f(z)}\right| \leq 1$. Moreover, by the chain rule, $\frac{d}{dz}\left(\frac{1}{f(z)}\right) = \frac{-f'(z)}{f(z)^2}$

and $\left|\frac{d}{dz}\left(\frac{1}{f(z)}\right)\right| \leq |f'(z)|$

so the derivative exists $\Rightarrow$ $F(z) = \frac{1}{f(z)}$ is entire. As $|F(z)| \leq 1$, $F$ is a bounded, entire function. Liouville's Theorem states $F(z)$ must be a constant that is non zero by hypothesis on $f$.