Problem 1.

i. Define what it means for a function $f : [a, b] \to R$ to be absolutely continuous.

ii. For a function $f \in AC[a, b]$, define the total variation $T_f(a, b)$. Prove that

$$T_f(a,b) \le \int_a^b |f'(s)|\, ds.$$

iii. A function $f : [a, b] \to R$ is Lipschitz if there exists a finite constant $M > 0$ so that for all $x, y \in [a, b]$,

$$|f(x) - f(y)| \le M|x - y|.$$

Prove that a function is Lipschitz if and only if

1. $f \in AC[a, b]$, and
2. $|f'(x)| \le M$ a.e. $x \in [a, b]$.

(i) $f: [a,b] \to \mathbb{R}$ is ac if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. for any finite collection of disjoint open intervals $\{(a_i, b_i)\}_{i=1}^N$ in $[a,b]$ with $\sum_{i=1}^N (b_i - a_i) < \delta$ we have $\sum_{i=1}^N |f(b_i) - f(a_i)| < \varepsilon$.

(ii) Let $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ be a partition of $[a,b]$. The variation of $f \in AC[a,b]$ with respect to P is $V_f^P[a,b] = \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$. The total variation is $T_f[a,b] = \sup_{P:[a,b]} V_f^P[a,b]$, where the supremum is taken over all partitions of $[a,b]$. If $f \in AC[a,b]$, f' exists a.e. and for any $x < y$, $x, y \in [a,b]$: $f(y) - f(x) = \int_x^y f'(s)\,ds$. Let P be a partition so $f(x_i) - f(x_{i-1}) = \int_{x_{i-1}}^{x_i} f'$ and summing over i: $V_f^P[a,b] = \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \le \int_a^b |f'|$. Since the right side is independent of P, it follows that $T_f[a,b] \le \int_a^b |f'|$.

(iii) $\Rightarrow$ Assume $f \in Lip[a,b]$. If $\{(a_i, b_i)\}_{i=1}^N$ is a finite collection of disjoint open intervals: $\sum_{i=1}^N |f(b_i) - f(a_i)| \le M \sum_{i=1}^N (b_i - a_i)$ by the Lipschitz cond. Given $\varepsilon > 0$ take $\delta < \varepsilon/M$ so if, in addition, the collection of disjoint intervals satisfies $\sum (b_i - a_i) < \delta$, we get $\sum |f(b_i) - f(a_i)| < \varepsilon \Rightarrow f \in AC[a,b]$. Since f' exists a.e. for any such x, $\lim_{h \to 0} \frac{|f(x+h) - f(x)|}{h} \le M \Rightarrow |f'(x)| \le M$ a.e.

$\Leftarrow$ Assume (1)-(2). For $a \le x < y \le b$, (1) implies $f(y) - f(x) = \int_x^y f'$, so $|f(y) - f(x)| \le \int_x^y |f'| \le M|y - x|$ by (2) so $f \in Lip[a,b]$. Note that we only used $|f'(x)| \le M$ a.e. x. ∎

Problem 2.

i. Suppose $E \subset R^d$. State what it means for a collection $\mathcal{B}$ of open balls to be a cover of E in the sense of Vitali.

ii. Let $E \subset R^d$ with $m_*(E) > 0$. For any $\epsilon > 0$, prove that there is an open set O containing E so that $m_*(O) \leq (1+\epsilon) m_*(E)$.

iii. Suppose $\mathcal{B}$ is a Vitali cover of $E \subset R^d$ with $0 < m_*(E) < \infty$. Apply Vitali I to show that for any $\epsilon > 0$ there exists a finite collection $\{B_1, \ldots, B_N\}$ of disjoint balls from $\mathcal{B}$ so that $m_*(E \setminus \cup_{j=1}^N B_j) \leq \epsilon$ and $\sum_{j=1}^N m_*(B_j) \leq (1+\epsilon) m_*(E)$. HINT: Given $\epsilon > 0$, choose an open set O with $E \subset O$ as in (ii.). By keeping only those elements of the Vitali cover contained in O the second condition is automatically satisfied.

i. $E \subset \mathbb{R}^d$. A collection $\mathcal{B}$ of balls is a Vitali cover of E if $\forall \eta > 0$ and $\forall x \in E$ $\exists B \in \mathcal{B}$ with $x \in B$ and $m(B) < \eta$.

ii. Recall that $m_*(E) = \inf\limits_{\mathcal{O} \supset E \text{ open}} m_*(\mathcal{O})$. Since $m_*(E) > 0$ for any $\varepsilon > 0$ $\exists$ open $\mathcal{O}_\varepsilon \supset E$ s.t. $m_*(\mathcal{O}) \leq m_*(E)(1+\varepsilon)$.

iii. Let $0 < m_* E < \infty$ and $\mathcal{B}$ be a Vitali cover of E. Recall Vitali I: For any finite collection $\mathcal{B} = \{B_1, \dots, B_N\}$ of balls $\exists$ finite subcollection $\{B_{i_1}, \dots, B_{i_k}\} \subset \mathcal{B}$ of disjoint balls s.t. $m\left(\bigcup_{j=1}^N B_j\right) \leq 3^d \sum_{\ell=1}^k m(B_{i_\ell})$.

Proof Given $\varepsilon > 0$ choose $\mathcal{O}_\varepsilon$ as in (ii). Pass to a Vitali cover $\tilde{\mathcal{B}}$ of E with all balls contained in $\mathcal{O}_\varepsilon$ (only small $\eta > 0$ is important). Since $m\mathcal{O}_\varepsilon < \infty$ $\exists K \subset \mathcal{O}_\varepsilon$ compact with $m_* K > \frac{1}{2} m_*(E)$. $\exists$ finite subcover of K in $\tilde{\mathcal{B}}$ to which we apply Vitali I: $\exists B_1, \dots, B_N \in \tilde{\mathcal{B}}$ disjoint with $\sum_{j=1}^{N_1} mB_j \geq \frac{1}{3^d} m(\bigcup B_j) = \frac{1}{2 \cdot 3^d} m_* E$. As in the proof of Vitali II, if $\sum^{N_1} mB_j > m_*(E) - \varepsilon \Rightarrow m_*(E \setminus \bigcup B_j) < \varepsilon$ so we're done; if not we repeat with $E_2 = E - \bigcup_{j=1}^{N_1} B_j$. The new collection of disjoint balls (N_2 of them) gives $\sum_{j=1}^{N_2} mB_j \geq \frac{2}{2 \cdot 3^d} m_*(E)$. If necessary, repeat M times so finally we have a collection of $N_1 + N_2 + \dots + N_M$ disjoint balls with $\sum_{j=1}^{N_M} mB_j \geq \frac{M}{2 \cdot 3^d} m_*(E)$. This is bigger than $m_* E - \varepsilon$ when M is s.t. $\varepsilon > m_*(E)\left(1 - \frac{M}{2 \cdot 3^d}\right)$. ■

Problem 3. Let $f \in L^1(R^d)$ and for any $a \in R^d$ define $f_a(x) = f(x-a)$.

i. Show that $f_a \in L^1(R^d)$ and that $\int_{R^d} f_a = \int_{R^d} f$. HINT: Use various approximations to f.

ii. Show that

$$\lim_{a\to 0} \int_{R^d} |f - f_a| = 0.$$

HINT: Use the fact that $C_0(R^d)$ (continuous functions with compact support) are dense in $L^1(R^d)$.

(i) $f_a(x) = f(x-a)$. 1) $E \subset \mathbb{R}^d$ m. $f = \chi_E \Rightarrow f_a = \chi_{E+a}$, $E+a$ m and $\int f_a = \int \chi_{E+a} = m(E+a) = m(E) = \int f$.

2) Simple fnc. $f = \sum c_j \chi_{F_j}$, F_j m. Then $f_a = \sum c_j \chi_{F_j+a}$ and
$$\int f_a = \sum c_j \int \chi_{F_j+a} = \sum c_j \int \chi_{F_j} = \int f.$$

3) bdd fnc w/ support of finite m. $\exists\ \varphi_n \to f$ s.t. $\lim \int \varphi_n = \int f$, φ_n simple. Then $\varphi_{n,a} \to f_a$ ptw. a.e. & $\varphi_{n,a}$ are simple.
$\lim\limits_n \int \varphi_{n,a} = \int f_a = \lim\limits_n \int \varphi_n = \int f$ by part (2).

4) $f \geq 0$ m & $f \in L^1$: $\int f = \sup\limits_{0 \leq g \leq f} \int g$, g as in (3).
The inequality $0 \leq g \leq f \Rightarrow 0 \leq g_a \leq f_a$ so $\sup\limits_{0 \leq g_a \leq f_a} \int g_a = \int f_a$
$= \sup\limits_{0 \leq g \leq f} \int g = \int f$.

5) $f = f_+ - f_-$ w/ $f_\pm$ as in (4). Then $f_a = f_{+,a} - f_{-,a}$ and
$\int f_a = \int f_{+,a} - \int f_{-,a} = \int f$, for $f \in L^1(\mathbb{R}^d)$.

(ii) $C_0(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ dense. Take $f \in L^1(\mathbb{R}^d)$. For $\varepsilon > 0$ $\exists\ g \in C_0(\mathbb{R}^d)$ s.t. $\int |f - g| < \varepsilon/3$. By part (i), $\int |g_a - f_a| < \varepsilon/3$. Since g is continuous $\lim\limits_{a\to 0} g_a = g$ and $g_a - g \to 0$. By the Bounded Conv. Th. $\lim\limits_{a\to 0} \int |g_a - g| = 0$, so $\exists\ a_0 > 0$ s.t. $a < a_0$ $\int |g_a - g| < \varepsilon/3$.

Then:
$$\int |f_a - f| \leq \int |f_a - g_a| + \int |g_a - g| + \int |g - f|$$
$$\leq \frac{2\varepsilon}{3} + \int |g_a - g| \leq \varepsilon.$$

$\Rightarrow \lim\limits_{a\to 0} \int |f_a - f| = 0.$ ∎

Problem 4. Suppose $f : E \to R^+$ is a nonnegative measurable function with $\int_E f < \infty$. Prove that for any $\epsilon > 0$ there exists a $\delta > 0$ so that if $A \subset E$ is measurable with $m(A) < \delta$, then $\int_A f < \epsilon$.

Proof: Set $E_N = \{x \in E \mid f(x) \leq N\}$.

$E_N \subset E$ is measurable.

Let $f_N \equiv \chi_{E_N} f$ so $f_N \nearrow f$ and by the Mon-tone Conv. Th. $\lim\limits_{N\to\infty} \int_E f_N = \int_E f$.

Given $\varepsilon > 0$ $\exists\, N_\varepsilon$ s.t. $\int_E (f - f_{N_\varepsilon}) < \varepsilon/2$.

For any $A \subset E$ measurable,

$$\int_A f \leq \int_E (f - f_{N_\varepsilon}) + \int_A f_{N_\varepsilon}$$

$$\leq \varepsilon/2 + m(A) \cdot N_\varepsilon .$$

Let $\delta > 0$ be s.t. $\delta N_\varepsilon < \varepsilon/2$. Then if $A \subset E$ is measurable & $m(A) < \delta$,

$$\int_A f \leq \varepsilon/2 + \delta N_\varepsilon < \varepsilon . \blacksquare$$

Problem 5. Let f_k be a nondecreasing sequence of measurable functions on a measurable set $E \subset R^d$, such that

$$\lim_k f_k(x) = f(x), \quad a.e. x \in E.$$

Suppose that $f_1 \in L^1(E)$. Prove that $f \in L^1(E)$ if and only if $\lim_k \int_E f_k < \infty$.

Given: $f_{k+1} \geq f_k$, $f_k \to f$ ptw a.e. and $f_1 \in L^1(E)$.

$\Rightarrow$ $f \in L^1(E)$. The nondecreasing property implies $f - f_k \geq 0$ for all k. Furthermore, $f - f_k \to 0$ a.e. Since $0 \leq f - f_k \leq f - f_1 \in L^1(E)$ (here we need $f_1 \in L^1(E)$) the Lebesgue Dominated Conv. th. implies

$$\lim_{k\to\infty} \int_E f_k = \int_E f < \infty$$

Note that $|f_k| \leq |f|$ need not be true think of $f_k < 0 \uparrow f = 0$ so one needs to adjust the seq. by f_1.

$\Leftarrow$ $\lim_k \int f_k < \infty$. We use again the fact that $f_k - f_1 \geq 0$ and $\lim_k (f_k - f) = f - f_1$ ptw. a.e. We can apply Fatou's lemma:

$$0 \leq \int \lim_{k\to\infty} (f_k - f_1) \leq \lim_k \int f_k - \int f_1 \quad (f_1 \in L^1(E))$$

$\Rightarrow$ $\int f$ exists so $f \in L^1(E)$. ∎