MA676 Spring 2009
Homework Problem Set #2
February 1, 2009

Assignment 2. Read sections 2-4 of Stein-Shakarchi (S^2). These problems are due Monday, 9 February 2009. (W-Z means the problems are from Wheeden-Zygmund).

1. Prove the very useful identity: For any $E \subset R^d$, and for any $\epsilon > 0$, there exists a countable cover of E by closed cubes Q_j so that

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \epsilon.$$

2. Give a second proof of Property 2 on page 17 of S^2 without using the observation 2 (that is, using directly the definition of outer measure).
3. (W-Z, page 48) Suppose that E_j is a sequence of subsets of R^d with $\sum_{j=1}^{\infty} m_*(E_j) < \infty$. Prove that $\limsup E_j$ and $\liminf E_j$ have measure zero.
4. (W-Z, page 48) If E_1 and E_2 are measurable sets, prove that $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.
5. Prove that outer measure is translation invariant. Then prove that if E is measurable, all its translates are measurable.
6. (W-Z, page 48) Prove that there exist disjoint sets E_j so that if $E = \cup_j E_j$, one has $m_*(E) < \sum_j m_*(E_j)$, where the inequality is strict. This proves the non-countable additivity of outer measure. For the construction, let $N \subset [0, 1]$ be nonmeasurable and such that all its rational translates are disjoint. Consider the sets obtained from the translates of N by rationals in $(0, 1)$.