Assignment 2. Read sections 2-4 of Stein-Shakarchi ($S^2$). These problems are due Monday, 9 February 2009. (W-Z means the problems are from Wheeden-Zygmund).

1. Prove the very useful identity: For any $E \subset \mathbb{R}^d$, and for any $\epsilon > 0$, there exists a countable cover of $E$ by closed cubes $Q_j$ so that

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \epsilon.$$

2. Give a second proof of Property 2 on page 17 of $S^2$ without using the observation 2 (that is, using directly the definition of outer measure).

3. (W-Z, page 48) Suppose that $E_j$ is a sequence of subsets of $\mathbb{R}^d$ with $\sum_{j=1}^{\infty} m_*(E_j) < \infty$. Prove that $\limsup E_j$ and $\liminf E_j$ have measure zero.

4. (W-Z, page 48) If $E_1$ and $E_2$ are measurable sets, prove that $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

5. Prove that outer measure is translation invariant. Then prove that if $E$ is measurable, all its translates are measurable.

6. (W-Z, page 48) Prove that there exist disjoint sets $E_j$ so that if $E = \bigcup_j E_j$, one has $m_*(E) < \sum_j m_*(E_j)$, where the inequality is strict. This proves the non-countable additivity of outer measure. For the construction, let $N \subset [0,1]$ be nonmeasurable and such that all its rational translates are disjoint. Consider the sets obtained from the translates of $N$ by rationals in $(0,1)$.