MA676 Spring 2009
Homework Problem Set #5
March 26, 2009

These problems are on the material in chapters 2 and 3 of Stein-Shakarchi. These problems are due Wednesday, 1 April 2009. Problem discussion Friday, 27 March at 4PM. (WZ means the problems are from Wheeden-Zygmund).

(1) Let $f \geq 0$ be integrable and define $E_a = \{x \in \mathbb{R}^d \mid f(x) > a\}$. Use the Chebychev inequality to prove that $\lim_{a\to\infty} am(E_a) = 0$.
(2) (WZ). If f, g are measurable on $\mathbb{R}^d$, then $h = fg$ is measurable on $\mathbb{R}^{2d}$.
(3) S^2, page 93, problem 19. For simplicity, assume $f \geq 0$ and $f \in L^1(\mathbb{R}^d)$. To get started, let $E_y = \{x \mid f(x) > y\}$ and define (as in WZ) $R(f, E) = \{(x, y) \in \mathbb{R}^{d+1} \mid x \in E, 0 \leq y \leq f(x),$ if $f(x) < \infty$, and $0 \leq y < \infty$, if $f(x) = \infty\}$. This is the region under the graph of f over $E \subset \mathbb{R}^d$. Show that $R(f, E)$ is meaurable and that

$$m(R(f,E)) = \int_E f, \text{ and } \int f = \int_{R(f,\mathbb{R}^d)} dx\ dy.$$

Finally, note that $\{x \in E \mid f(x) \geq y\} = \{x \in E \mid (x, y) \in R(f, E)\}$ and that $m(E_y) = m(\{x \in E \mid f(x) \geq y\}$ a.e. y. To conclude, use Tonelli's Theorem to prove that

$$\int f = \int_0^\infty m(E_y)\ dy.$$

(4) Use Egorov's Theorem to prove the Bounded Convergence Theorem.
(5) (WZ) If $f \in L^1([0, 1])$, then $x^k f(x) \in L^1([0, 1])$ and $\lim_{k\to\infty} \int_0^1 x^k f(x) = 0$.