These problems are on the material in chapters 2 and 3 of Stein-Shakarchi. These problems are due Wednesday, 1 April 2009. Problem discussion Friday, 27 March at 4PM. (WZ means the problems are from Wheeden-Zygmund).

1. Let \( f \geq 0 \) be integrable and define \( E_a = \{ x \in \mathbb{R}^d \mid f(x) > a \} \). Use the Chebychev inequality to prove that \( \lim_{a \to \infty} m(E_a) = 0 \).

2. (WZ) If \( f, g \) are measurable on \( \mathbb{R}^d \), then \( h = fg \) is measurable on \( \mathbb{R}^{2d} \).

3. \( S^2 \), page 93, problem 19. For simplicity, assume \( f \geq 0 \) and \( f \in L^1(\mathbb{R}^d) \).
   - To get started, let \( E_y = \{ x \mid f(x) > y \} \) and define (as in WZ) \( R(f, E) = \{(x,y) \in \mathbb{R}^{d+1} \mid x \in E, 0 \leq y \leq f(x), \text{ if } f(x) < \infty, \text{ and } 0 \leq y < \infty, \text{ if } f(x) = \infty \} \). This is the region under the graph of \( f \) over \( E \subset \mathbb{R}^d \).
   - Show that \( R(f, E) \) is measurable and that \( m(R(f, E)) = \int_E f \), and \( \int f = \int_{R(f, \mathbb{R}^d)} dx \ dy. \)
   - Finally, note that \( \{ x \in E \mid f(x) \geq y \} = \{ x \in E \mid (x, y) \in R(f, E) \} \) and that \( m(E_y) = m(\{ x \in E \mid f(x) \geq y \} \) a.e. \( y \). To conclude, use Tonelli’s Theorem to prove that \( \int f = \int_0^\infty m(E_y) \ dy. \)

4. Use Egorov’s Theorem to prove the Bounded Convergence Theorem.

5. (WZ) If \( f \in L^1([0,1]) \), then \( x^k f(x) \in L^1([0,1]) \) and \( \lim_{k \to \infty} \int_0^1 x^k f(x) = 0. \)