MA676 Problem Set 1 - Solutions
Spring 09

(PS1-1)

1. $f: [0,1] \to \{0,1\}$ defined by $f(x) = 1$, $x \in \mathbb{Q} \cap [0,1]$, $f(x) = 0$ otherwise. By contradiction: suppose f is cont. at $x_0 \in (0,1)$ so for any $\varepsilon > 0$ $\exists$ $\delta_\varepsilon(x_0) > 0$ s.t. $|x - x_0| < \delta_\varepsilon(x_0)$ then $|f(x) - f(x_0)| < \varepsilon$. If $x_0 \in \mathbb{Q} \cap [0,1]$ $f(x_0) = 1$ and $\exists$ $y \in B_{\delta_\varepsilon(x_0)}(x_0)$, $y \in [0,1] \setminus \mathbb{Q} \cap [0,1]$ so $f(y) = 0$ so $|f(x_0) - f(y)| = 1 \nless \varepsilon$, contradition. If $x_0 \in \mathbb{Q}^c \cap [0,1]$ $f(x_0) = 0$ and one can choose $y \in B_{\delta_\varepsilon(x_0)}(x_0)$ s.t. $f(y) = 1$ so again $|f(x_0) - f(y)| = 1 \nless \varepsilon$, contradiction. Finally, for $x_0 = 0$ or $x_0 = 1$, take $B_{\delta_\varepsilon(0)}(0) \cap [0,1]$ and $B_{\delta_\varepsilon(1)}(1) \cap [0,1]$ and the same argument works. ∎

2. We know $\mathcal{O} = \bigcup_{j=1}^{\infty} I_j$ where I_j is an open interval and $I_j \cap I_k = \emptyset$ if $j \neq k$. Suppose $\mathcal{O} = \bigcup_{\ell=1}^{\infty} J_\ell$ is another such representation. Each interval is written as $I_j = (a_j, b_j)$, $a_j < b_j$ and $J_\ell = (c_\ell, d_\ell)$ with $c_\ell < d_\ell$. We order the a_j's and c_ℓ's so $a_1 < a_2 < \cdots$ $c_1 < c_2 < \cdots$ then necessarily (due to the fact that the intervals are disjoint) $b_j < a_{j+1}$ and $d_\ell < c_{\ell+1}$. Begin with I_1 & J_1! $a_1 = c_1$ or otherwise, if $a_1 < c_1$, $\bigcup J_\ell$ doesn't contain all the points of $\mathcal{O}$. Next, $b_1 = d_1$ for if not, say $b_1 < d_1$, and either $a_2 < d_1$, in which case there are pts of $\mathcal{O}$ not in $\bigcup I_j$, or $a_2 > d_1$ (so $b_1 < d_1 < a_2$) and again, there are pts of $\mathcal{O}$ not in $\bigcup I_j$ or $a_2 = d_1$ so $b_1 \in \mathcal{O}$ again a contradiction. So suppose we have $I_m = J_m$, $m = 1, \ldots, n-1$. and look at $I_n = (a_n, b_n)$ & $J_n = (c_n, d_n)$. The same reasoning shows $a_n = c_n$ & $b_n = d_n$. So by induction $I_m = J_m$ $\forall m$. Thus the 2 representations are the same.

3. (WZ pg 13) Let $K_1, K_2 \subset \mathbb{R}^d$ be nonempty disjoint compact sets $K_1 \cap K_2 = \emptyset$. The distance between them is $d(K_1, K_2) = \inf_{\substack{x \in K_1 \\ y \in K_2}} |x - y|$. Suppose $d(K_1, K_2) = 0$ so $\exists$ seq $x_n - y_n$, $x_n \in K_1$,

$y_n \in K_2$ with $|x_n - y_n| \to 0$. Since $\{x_n\} \subset K_1$ & K_1 is compact $\exists\ x_{n_j} \to x_0 \in K_1$. Similarly $\exists\ \{y_{n_j}\} \subset K_2$ with $y_{n_j} \to y_0 \in K_2$. Then $|x_{n_j} - y_{n_j}| \to 0$ so $x_0 = y_0 \in K_1 \cap K_2$, a contradiction. This means $d(K_1, K_2) > 0$. ■

$|x_0 - y_{n_j}| \leq |x_0 - x_{k_j}| + |x_{k_j} - y_{k_j}| \to 0$

4. (WZ pg 12) i) $\overline{\lim}\, E_j = \bigcap_{j=1}^{\infty} \left(\bigcup_{k=j}^{\infty} E_k \right)$. If $x \in \overline{\lim}\, E_j$ then $x \in \bigcup_{k=j}^{\infty} E_k$ for all j. If x belonged to only finitely many sets $E_1, \dots, E_{j_0}$ this means $x \notin \bigcup_{k=j_0+1}^{\infty} E_k$, a contradiction.

Conversely, if x belongs to infinitely many E_ℓ's then $x \in \bigcup_{k=j}^{\infty} E_k$ for all j $\Rightarrow$ $x \in \overline{\lim}\, E_j$.

ii) $\underline{\lim}\, E_j = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} E_k$, then $x \in \underline{\lim}\, E_j$ if $x \in \bigcap_{k=j}^{\infty} E_k$ any $\underline{\text{one}}$ j

so x belongs to all but finitely-many E_j. Clearly, if $x \in E_\ell$ for all but finitely many ℓ, $E_1, \dots, E_{\ell_0}$, $x \in \bigcap_{k=\ell_0+1}^{\infty} E_k$ so $x \in \underline{\lim}\, E_j$.

5. (WZ pg 13) Let $E_k = \begin{cases} [-\frac{1}{k}, 1] & k \text{ odd} \\ [-1, \frac{1}{k}] & k \text{ even} \end{cases}$. $\bigcap_k E_{2k+1} = [0,1]$ & $\bigcap E_{2n} = [-1,0]$

$\underline{\lim}\, E_n = \{0\}$ only point in $\underline{\text{all}}$ E_j for some j onward as $\bigcap_{k \geq j} E_k = \{0\}$.

$\overline{\lim}\, E_n = [-1,1]$ as any pt in $[-1,1]$ belongs to infinitely many E_j. since $\bigcup_{k \geq j} E_k = [-1,1]$

6. Let K_j be a decr seq $K_j \supset K_{j+1}$ (strict) of nonempty compact sets. Let $x_j \in K_j \setminus K_{j+1}$, so $x_j \in K_1$ $\forall j$. As K_1 is compact $\exists$ convergent subseq. $\{x_{k_j}\}$ and $x_{k_j} \to x_0 \in K_1$. Claim: $x_0 \in \bigcap K_j$. If not, $\exists\ J_0$ s.t. $x_0 \notin K_{J_0}$ and $x_0 \notin \bigcap_{m \geq J_0} K_m$ so $d(x_0, K_{J_0}) > 0$ by

problem 3. But all but finitely many x_j's belong to $\bigcap_{m \geq J_0} K_m$ and consequently all but finitely many $x_{k_j} \in \bigcap_{m \geq J_0} K_m$ so x_0 can't be their limit), a contradiction.

So, $x_0 \in \bigcap_{j \geq 1} K_j$ and it is nonempty. ∎

7. Example: K_j noncompact closed. $K_j \supset K_{j+1}$ but $\bigcap K_j = \phi$

$K_j \equiv \{ (x_1, \ldots, x_{n-1}, y) \mid y \geq j \}$ closed half-space

$K_{j+1} \subset K_j \ \forall j$ but $\bigcap K_j = \phi$ (if $x \in \bigcap K_j$, $x_n \geq j \ \forall j$.).