MAT 76 Problem Set 1 - Solutions
Spring '99

1. \( f: [0, 1] \rightarrow \{0, 1\} \) defined by \( f(x) = 1 \), \( x \in \mathbb{Q} \cap [0, 1) \), \( f(x) = 0 \) otherwise.

By contradiction: suppose \( f \) is cont. at \( x_0 \in (0, 1) \) so for any \( \varepsilon > 0 \) \( \delta_\varepsilon \) such that \( |f(x) - f(x_0)| < \varepsilon \), if \( x_0 \in \mathbb{Q} \cap [0, 1) \) \( f(x_0) = 1 \) and there exists \( y \in \mathbb{Q} \cap [0, 1) \) \( f(y) = 0 \) so \( |f(y) - f(x_0)| = 1 \), contradiction. If \( x_0 \in \mathbb{Q}^c \cap [0, 1) \) \( f(x_0) = 0 \) and one can choose \( y \in (x_0 - \delta, x_0) \) s.t. \( f(y) = 1 \) so again \( |f(y) - f(x_0)| = 1 \), contradiction. Finally, for \( x_0 = 0 \) or \( x_0 = 1 \), take \( \delta = 1/N_0 \) and the same argument works.

2. We know \( \Omega = \bigcup_{i=1}^\infty \Omega_i \) where \( \Omega_i \) is an open interval and \( \Omega_i \cap \Omega_j = \emptyset \) if \( i \neq j \). Suppose \( \Omega = \bigcup_{i=1}^\infty \Omega_i \) is another such representation. Each interval is written as \( \Omega_i = (a_i, b_i) \), \( a_i < b_i \) and \( \Omega_j = (c_j, d_j) \), \( c_j < d_j \). We order the \( a_i \)'s and \( c_j \)'s so \( a_1 < a_2 < \ldots < c_1 < c_2 < \ldots \) then necessarily (due to the fact that the intervals are disjoint) \( b_1 < a_{j+1} \) and \( d_1 < c_{j+1} \). Begin with \( \Omega_i \) and \( \Omega_j \) \( a_i = c_j \) or otherwise, if \( a_i < c_j \), \( \Omega_i \) doesn't contain all the points of \( \Omega \). Next, \( b_1 = d_1 \) for it not, say \( b_1 < d_1 \), and either \( a_2 < d_1 \), in which case there are pts of \( \Omega \) not in \( \Omega_i \), or \( a_2 > d_1 \) (so \( b_1 < d_1 < a_2 \)) and again, there are pts of \( \Omega \) not in \( \Omega_i \) or \( a_2 = d_1 \), so \( b_1 \in \Omega \) again a contradiction. So suppose we have \( \Omega = \bigcup_{i=1}^{m-1} \Omega_i \) and \( \Omega = (a_i, b_n) \) \( \Omega = (c_k, d_n) \). The same reasoning shows \( a_n = c_n \) and \( b_n = d_n \). So by induction, \( \Omega = \bigcup_{i=1}^{m} \Omega_i \). Thus the 2 representations are the same.

3. (WZ pg. 13) Let \( K_1, K_2 \subset \mathbb{R}^d \) be nonempty disjoint compact subsets \( K_1 \cap K_2 = \emptyset \). The distance between them is \( d(K_1, K_2) = \inf \{x - y : x \in K_1, y \in K_2\} \). Suppose \( d(K_1, K_2) > 0 \) so for \( \xi \) seq. \( x_\xi - y_\xi \), \( y_\xi \in K_2 \).
\[ y_n \in k_2 \text{ with } |x_n - y_n| \to 0. \text{ Since } k_1 \text{ is compact, } \exists x_n \to x_0 \in k_1. \text{ Similarly } \exists y_n \to y_0 \in k_2. \text{ Then } |x_n - y_n| \to 0 \text{ so } x_0 = y_0 \in k_1 \cap k_2, \text{ a contradiction. This means } d(k_1, k_2) > 0. \]

4. (WZ pg 12) i) \( \overline{\lim E_j} = \bigcap_{j=1}^{\infty} (U_k \bigcup_{j=1}^{k} E_j) \). If \( x \in \overline{\lim E_j} \) then \( x \in U_k \bigcup_{j=1}^{\infty} E_j \) for all \( j \). If \( x \) belonged to only finitely many sets \( E_1, \ldots, E_j \), this means \( x \notin \bigcup_{j=1}^{\infty} E_j \), a contradiction.

Conversely, if \( x \) belongs to infinitely many \( E_j \)'s then \( x \in \bigcup_{j=1}^{\infty} E_j \) for all \( j \). x \( \in \overline{\lim E_j} \).

ii) \( \overline{\lim E_j} = \bigcap_{j=1}^{\infty} E_j \), then \( x \in \overline{\lim E_j} \) if \( x \notin \bigcap_{k=j}^{\infty} E_k \) for any one \( j \).

so \( x \) belongs to all but finitely many \( E_j \)'s. Clearly, if \( x \in E_j \) for all but finitely many \( j \), \( E_1, \ldots, E_{j_0} \), \( x \in \bigcap_{k=j_0+1}^{\infty} k \) so \( x \in \overline{\lim E_j} \).

5. (WZ pg 13) Let \( E_k = \begin{cases} [-\frac{1}{k+1}, \frac{1}{k+1}] \text{ if } k \text{ odd} \\ [-\frac{1}{k}, \frac{1}{k}] \text{ if } k \text{ even} \end{cases} \). \( \bigcap_{k=1}^{\infty} E_k = \{0\} \text{ and } \bigcap_{k=1}^{2n} E_k = [-1/n, 1/n] \). 

\( \overline{\lim E_n} = \{0\} \) only point in all \( E_j \) for some \( j \) onward as \( \bigcap_{k=1}^{2n} E_k = \{0\} \).

\( \overline{\lim E_n} = [-1, 1] \) as any \( p \) in \( [-1, 1] \) belongs to \( \bigcap_{k=1}^{\infty} E_k = \{0\} \).

since \( \bigcap_{k=1}^{\infty} E_k = [-1, 1] \)

6. Let \( k_j \) be a decr seq \( k_j \geq k_{j+1} \) (strict) of nonempty compact subs. Let \( x_j \in k_j \setminus k_{j+1} \), so \( x_j \in k_1 \forall j \). As \( k_1 \) is compact \( \exists \) convergent subseq. \( \{x_{j_k}\} \) \( \quad \) and \( x_k \to x_0 \in k_1 \). Claim: \( x_0 \notin k_j \) \( \forall j \). If not, \( \exists J_0 \) s.t. \( x_0 \in k_{J_0} \) and \( x_0 \notin k_{J_0} \cap k_{J_0} \) so \( d(x_0, k_{J_0}) > 0 \) by
problem 3. But all but finitely many \( x_j \)'s belong to \( \cap K_m \) and 
consequently all but finitely many \( x_k, j \in \cap K_m \) so \( x_0 \) can't be
their limit, a contradiction.
So, \( x_0 \in \cap K_j \) and it is nonempty.

7. Example: \( K_j \) noncompact \( K_j \supset K_{j+1} \) but \( \cap K_j \neq \emptyset \)
closed.

\[ K_j = \{ (x_1, ..., x_{n-1}, y) \mid y \geq j \} \quad \text{closed half-space} \]

\[ K_{j+1} \subset K_j \forall j \] but \( \cap K_j = \emptyset \) (if \( x \in \cap K_j \),
\[ x_n > j \forall j \). \]