Spring 09

MA 676 Solutions to PS 2

1. $\forall \varepsilon > 0$ $\exists$ countable cover by cld cubes of E s.t. $\sum_{j=1}^{\infty} |Q_j| \le m_*(E) + \varepsilon$.

(1)

Pf $E \subset \mathbb{R}^d$, let $\mathcal{C}(E) \equiv$ collection of C^4's for E. Then $m_*(E) = \inf_{\mathcal{C}(E)} \sum_j |Q_j|$. If the condition did not hold true then $\exists$ $\varepsilon_0 > 0$ s.t. $\forall \{Q_j\} \in \mathcal{C}(E)$

$$\sum_j |Q_j| > m_*(E) + \varepsilon_0 .$$

But this means $\inf_{\mathcal{C}(E)} \sum |Q_j| - \varepsilon_0 > m_*(E)$

a contradiction. ∎

2. If $m_*(E) = 0$ then E is m. (If $F \subset E$, then $0 \le m_*(F) \le m_*(E) = 0 \Rightarrow m_*(F) = 0$.). Pf. By problem 1, for each $\varepsilon > 0$ $\exists$ $\{Q_j\} \in \mathcal{C}(E)$ s.t. $0 \le \sum |Q_j| \le m_*(E) + \varepsilon/2 = \varepsilon/2$. For each j let $\tilde{Q}_j \supset Q_j$ be an open cube with $|\tilde{Q}_j| \le |Q_j| + \varepsilon/2^{j+1}$. Then setting $\mathcal{O}_\varepsilon = \bigcup_j \tilde{\mathcal{O}}_j$ open, containing E,

$$\begin{aligned} m_*(\mathcal{O}_\varepsilon - E) &\le m_*(\mathcal{O}_\varepsilon) \quad \text{monotonicity} \\ &\le \sum_{j=1}^{\infty} |\tilde{\mathcal{O}}_j| \le \sum_{j=1}^{\infty} |\mathcal{O}_j| + \varepsilon \sum_{j=1}^{\infty} \frac{1}{2^{j+1}} \\ &\le \varepsilon \end{aligned}$$

showing that E is m.

3. Suppose $\{E_j\}$ s.t. $\sum m_*(E_j) < \infty$. Then $\underline{\lim}\, E_j$ & $\overline{\lim}\, E_j$ have m zero.

Pf By defn. $\overline{\lim}\, E_j = \bigcap_{j=1}^{\infty} \left(\bigcup_{k=j}^{\infty} E_j \right)$. Since $\sum_{j=1}^{\infty} m_*(E_j) < \infty$

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given $\varepsilon > 0$ $\exists N_\varepsilon$ s.t. $\sum_{j=N_\varepsilon}^{\infty} m_*(E_j) < \varepsilon$. Then by monotonicity and sub-additivity:

$$m_*(\overline{\lim} E_j) \le m_*\left(\bigcup_{j=N_\varepsilon}^{\infty} E_j\right) \le \sum_{j=N_\varepsilon}^{\infty} m_*(E_j) < \varepsilon$$

for any $\varepsilon > 0$ $\Rightarrow$ $m_*(\overline{\lim} E_j) = 0$. Similarly, $\underline{\lim}\, E_j = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j$

so a $\underline{\lim}\, E_j \subset \overline{\lim}\, E_j$ we get (subset of m zero is m & m 0) that it has m zero.

$= 0$ so $\underline{\lim}\, E_j < \varepsilon$ $\forall \varepsilon \Rightarrow \underline{\lim}\, E_j = 0$. ∎

4. E_1, E_2 m then $m(E_1 \cup E_2) = m(E_1) + m(E_2)$.

<u>Pf</u> Write $E_1 \cup (E_2 - E_1) = E_1 \cup E_2$

and $E_2 = E_1 \cap E_2 \cup (E_2 - E_1)$

All sets are measurable & disjoint so

$$mE_1 + m(E_2 - E_1) = m(E_1 \cup E_2)$$
$$mE_2 = m(E_1 \cap E_2) + m(E_2 - E_1)$$

Add these to get

$$mE_1 + mE_2 = m(E_1 \cup E_2) + m(E_1 \cap E_2). \blacksquare$$

5. $E \subset \mathbb{R}^d$ then $m_*(E) = \inf_{\mathcal{O} \supset E \text{ open}} m_*(\mathcal{O})$. $h \in \mathbb{R}^d$ set $E_h = E + h$.

a) If $\mathcal{O} \supset E$ open then $\mathcal{O} + h \supset E_h$ open. Since $\mathcal{O} = \sum Q_j$ almost disjoint cubes so $m_*(\mathcal{O}) = \sum |Q_j|$. Since $|Q_j + h| = |Q_j|$ we get $m_*(\mathcal{O}+h) = m_*(\mathcal{O})$ $\forall \mathcal{O}$ open, $h \in \mathbb{R}^d$. Note $\mathcal{O} \supset E$ open iff $\mathcal{O}+h \supset E_h$ open so $m_*(E_h) = \inf_{\mathcal{O} \supset E_h \text{ open}} m_*(\mathcal{O}) = \inf_{\mathcal{O}+h \supset E \text{ open}} m_*(\mathcal{O}+h) = m_*(E)$. This shows m_* is transl. inv.

b) If $E \subset \mathbb{R}^d$ m, then $\forall \varepsilon > 0$ $\exists \mathcal{O}_\varepsilon$ s.t. $m_*(\mathcal{O}_\varepsilon - E) \le \varepsilon$, so $\mathcal{O}_\varepsilon + h \supset E_h$ and $m_*((\mathcal{O}_\varepsilon + h) - E_h) = m_*(\mathcal{O}_\varepsilon - E) \le \varepsilon \Rightarrow E_h$ m and $mE_h = mE$.