Spring 2009

3-1

MA 676 Problem Set 3 Solutions

Problem 1 For any $A \subset \mathbb{R}^d$ $\exists$ G_δ $G \supset A$ s.t. $m_* A = m_* G$.

Proof For any $\varepsilon > 0$, the defn of outer m. as

$m_* A = \inf_{\mathcal{O} \supset A \text{ open}} m_*(\mathcal{O})$, implies that $\exists\ \mathcal{O}_\varepsilon \supset A$ open

with $m_* \mathcal{O}_\varepsilon \le m_* A + \varepsilon$. Set $\varepsilon = \frac{1}{n}$ and form

$G = \bigcap_n \mathcal{O}_{\frac{1}{n}}$. G is a G_δ-set, $G \supset A$ since each $\mathcal{O}_{\frac{1}{n}} \supset A$, and

$$m_* A \le m_* G \le m_* \mathcal{O}_m \le m_* A + \frac{1}{m} \quad \forall m$$

This implies that $m_* G = m_* A$. ∎

Problem 2

pg 37. 1. Show that the ternary Cantor set C is totally disconnected & perfect.

Pf (i) Totally disconnected. $x \neq y \in C$ $\exists$ $k \in \mathbb{N}$ s.t. $|x-y| > \frac{1}{3^k}$. Since component C_k consists of 2^k intervals of length $\frac{1}{3^k}$, x, y are in different intervals in C_k so $\exists$ an open interval of length $\frac{1}{3^k}$, (a,b) such that $x < a$ and $y > b$. Any $z \in (a,b)$ satisfies $x < z < y$.

(ii) Suppose C isn't perfect and $x_0 \in C$ is isolated. As $x_0 \in C_k$ let $[a_k, b_k]$ be the interval containing x_0: $x_0 \in [a_k, b_k]$, $b_k - a_k = \frac{1}{3^k}$, so either $\lim_{k\to\infty} a_k = x_0$ or $\lim_{k\to\infty} b_k = x_0$ so as the endpts of C_n belong to C every ε-ball around x_0 contains a point of C, contradicting the hypothesis that x_0 is isolated.

Problem 2: #2 on Cantor Sets, pg 38.

Ternary expansion: $x \in [0,1]$. Let (a_j) be the coefficients of $\sum_{k=1}^{\infty} a_k 3^{-k}$.

If $x = 0$, $(a_j) = (0)$. If $x = 1$, $(a_j) = (2)$ since $\sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1}{2}$.

So for $x \in (0,1)$, take $\begin{cases} a_1 = 0 & 0 < x < 1/3 \\ a_1 = 1 & \frac{1}{3} \le x < 2/3 \\ a_1 = 2 & 2/3 \le x < 1 \end{cases}$

$x_1 = x - a_1/3$, so $0 \le x_1 < \frac{1}{3}$. Set $\begin{cases} a_2 = 0 & 0 \le x_1 < \frac{1}{3^2} \\ a_2 = 1 & \frac{1}{3^2} \le x_1 < 2/3^2 \\ a_2 = 2 & 2/3^2 \le x_1 < 1/3 \end{cases}$

$x_2 = x_1 - a_2/3^2$

$\vdots$

$x_k = x_{k-1} - a_k/3^k$ and define a_k accordingly.

Claim: $x \in (0,1)$ has a unique rep unless $x = q/3^k$ for $1 \le q \le 3^k$ and $k \in \mathbb{N}$. Any $x \in (0,1)$ with a finite ternary expansion $(a_1, \dots, a_m)$ all $a_n = 0$, $n \ge m+1$, can be written in this form and, conversely, any $\frac{q}{3^k}$ can be written as a finite ternary expansion. If x has a finite ternary expansion then it has a rep with an infinite number of terms. Since $\frac{1}{3^k} = \sum_{j=k+1}^{\infty} \frac{2}{3^j}$ if $a_k = 1$ we can replace this with $(0, \dots, 0, \underset{k^{th}\text{ place}}{0}, 2, 2, 2 \dots)$.

a) Cantor Set: $x \in \mathcal{C}$ iff x has a rep. with each a_j either 0 or 2.

Pf $\Rightarrow$ $x \in \mathcal{C} = \bigcap_{n \ge 1} \mathcal{C}_n$ so $x \in \mathcal{C}_n$ $\forall k$. $x \in \mathcal{C}_1 \Rightarrow x < \frac{1}{3}$ or $x > 2/3$ or x is an endpt of $\mathcal{C}_1$ so $a_1 = 0$ or 2 or $x = q/3$, $q = 0, 1, 2, 3$ in which we have a representation with only 0 or 2. $x \in \mathcal{C}_2$ means $0 < x < \frac{1}{3^2}$ or $\frac{2}{3^2} < x < \frac{1}{3}$ or an endpt. So $a_2 = 0$ or 2 continuing by induction, the rep of x is in terms of 0 or 2.

$\Leftarrow$ If $x = (a_1, a_2, \ldots)$ with a_k the first one. Then the previous a_j, $j<k$ are all zeros meaning x is in an excluded $1/3^k$ interval or, if no other $a_j \neq 0$, an endpt. that can be expressed in terms of 0 & 2. If a previous a_j, $j<k$, is 2, x is again in an excluded interval at level k. ∎

(b) $F(x) = \sum \frac{b_k}{2^k}$ with $x = \sum \frac{a_k}{3^k}$ and $b_k = \frac{a_k}{2}$.

Every $x \in \mathcal{C}$ has a unique rep with $a_k \in \{0,2\}$ so F is well-def. For continuity, let $x_0 \in \mathcal{C}$. Given $\varepsilon > 0$ choose N_ε s.t. $\sum_{j=N_\varepsilon}^{\infty} \frac{1}{2^j} < \varepsilon$. Then taking $\delta < \frac{1}{3^{N_\varepsilon}}$ if $x \in \mathcal{C}$ & $|x - x_0| < \delta$ then $x, x_0 \in \mathcal{C}_{N_\varepsilon}^{(i)}$ (i^{th} subinterval of $\mathcal{C}_{N_\varepsilon}$) and $a_j(x) = a_j(x_0)$ $j = 1, \ldots, N_\varepsilon - 1$. Then

$$|F(x) - F(x_0)| \leq \sum_{j=N_\varepsilon}^{\infty} \frac{1}{2^j} < \varepsilon,$$ proving continuity at $x_0 \in \mathcal{C}$.

Check $0 = (0, \ldots)$ $(a_j = 0)$ so $F(0) = 0$

$1 = (2, 2, \ldots)$ so $F(1) = 1$ since $\sum_{k=1}^{\infty} \frac{2}{3^k} = 1$

(c) If $x \in [0,1]$, x has a ! binary rep. (b_k), $b_k \in (0,1)$ so $x = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$. Clearly $F(x) = (b_k(x)) \in [0,1]$.

Conversely if $y \in [0,1]$ let $(b_k(y))$ be its binary rep. then $2b_k(y) \equiv a_k \in \{0,2\}$ and $x = \sum \frac{a_k}{3^k} \in \mathcal{C}$ s.t. $F(x) = y$ so F is surjective. F isn't injective since if x satisfies $\frac{1}{3} \leq x < \frac{2}{3}$, $F(x) = F(\frac{1}{3})$.

(d) $F: [0,1] \to [0,1]$ & continuous. If $(a,b) \subset \mathcal{C}^c$, $b - a \leq \frac{1}{3^k}$ for some k. Because each endpt. has $a_j = 2$ or 0 and because of the summability properties of $\sum \frac{1}{2^j}$ we get $F(a) = F(b)$!

Problem 3

pg 43 #25. E m if $\forall \varepsilon > 0$ $\exists$ cld $F \subset E$ $\ni$ $m_*(E-F) \le \varepsilon$.
Prove equivalence of this defn. & one used in class - let's say E is o-m if the open set defn applies & c-m if the closed set one does.

(1) Suppose E is o-m. We know E^c is o-m. For any $\varepsilon > 0$ $\exists$ $\mathcal{O}_\varepsilon$ open, $E^c \subset \mathcal{O}_\varepsilon$ with $m_*(\mathcal{O}_\varepsilon - E^c) \le \varepsilon$. Then $F = \mathcal{O}_\varepsilon^c$ is closed & $E \supset F$ & $m_*(E-F) \le m_*(\mathcal{O}_\varepsilon - E^c) \le \varepsilon$ so E is c-m.

(2) Suppose E is c-m. For any $\varepsilon > 0$ $\exists$ $F_\varepsilon \subset E$ cld and $m_*(E-F_\varepsilon) \le \varepsilon$. This implies E^c is open measurable since $F_\varepsilon^c \supset E^c$ and is open. Then we know E is o-m.

26. $A \subset E \subset B$, A & B m of finite m. If $m(A) = m(B)$ then E is m. Pf Simply note that $B = A \cup (B-A)$ and since A & B are m so is $B-A$. By finite additivity, $mB = mA + m(B-A) = mA$ so $m(B-A) = 0$. This means $E-A \subset B-A$ is measurable and has measure zero. As $E = A \cup (E-A)$, E is the union of 2 m sets and therefore m.
(subsets of sets of m zero are m & have m zero)

Problem 4 Inner measure. $m_i(A) \equiv \sup_{F \subset A \text{ cld}} m(F)$. Recall $m_*(A) = \inf_{\mathcal{O} \supset E \text{ open}} m_*(\mathcal{O})$

(i) $m_i(E) \le m_*(E)$. By monotonicity, if $F \subset E \subset \mathcal{O}$, F cld & $\mathcal{O}$ open $m_*(F) \le m_*(E) \le m_*(\mathcal{O})$. Taking sup over all $F \subset E \Rightarrow$
$m_i(E) \le m_*(E) \le m_*(\mathcal{O})$.
so $m_i(E) \le m_*(E)$.

(ii) If $m_*(E) < \infty$ then E m iff $m_i(E) = m_*(E)$.
Pf. $\Rightarrow$ E m then by Th. 3.4 (ii) $\forall \varepsilon > 0$ $\exists$ cld $F_\varepsilon \subset E$ s.t. $m(E-F_\varepsilon) < \varepsilon$. Then by additivity and $mE = m_*(E) < \infty$,

$mE \le mF_\varepsilon + \varepsilon$ since $mF_\varepsilon \le \sup_{F \subset E \text{ cld}} m_* F = m_i E < m_* E < \infty$
we get

$$mE \le m_i E + \varepsilon \ \forall \varepsilon.$$ Combined with

$m_i E \le mE$ from (i), it follow that $m_i E = mE$. ∎

$\Leftarrow$ Suppose $m_* E < \infty$ and $m_i E = m_* E$. Given $\varepsilon > 0$, $\exists$ cld $F_\varepsilon \subset E$ s.t.

$$m_* F_\varepsilon \ge m_* E - \varepsilon/2$$

and, by defn of m_*, $\exists\, \mathcal{O}_\varepsilon \supset E$ open with

$$m_* \mathcal{O}_\varepsilon \le m_* E + \varepsilon/2.$$

As $\mathcal{O}_\varepsilon$ & F_ε are <u>both</u> measurable,

$$m\mathcal{O}_\varepsilon = mF_\varepsilon + m(\mathcal{O}_\varepsilon - F_\varepsilon)$$

and

$$m_* E - \varepsilon/2 \le m_* \mathcal{O}_\varepsilon = mF_\varepsilon + m(\mathcal{O}_\varepsilon - F_\varepsilon) \le m_* E + \varepsilon/2$$

$$\therefore 0 \le (mF_\varepsilon - m_* E + \varepsilon/2) + m(\mathcal{O}_\varepsilon - F_\varepsilon) \le \varepsilon$$

$\Rightarrow m(\mathcal{O}_\varepsilon - F_\varepsilon) \le \varepsilon$. Now $E - F_\varepsilon \subseteq \mathcal{O}_\varepsilon - F_\varepsilon$
so monotonicity of outer m. implies

$$0 \le m_*(E - F_\varepsilon) \le \varepsilon$$

By the criteria of problem 3, we conclude E is m.