2. Property 6: If \( f \) is a function a.e. \( g \) on \( \mathbb{E} \), then \( g \) is measurable. Proof: Let \( E \) be the set for which \( f(x) \neq g(x) \). By hypothesis, \( |x| < a \) for each \( x \in E \), so \( \{x \mid f(x) < a\} = \bigcup_{x \in E} \{x \mid g(x) < a\} \). The union of two measurable sets is measurable.

3. Let \( \{f_n\} \) be a sequence of \( m \) functions on \([0,1]\) with \( |f_n(x)| < \infty \) a.e. \( x \in [0,1] \). If \( \exists \{c_n\}, c_n \geq 0 \) s.t. \( f_n(x)/c_n \to 0 \) a.e. \( x \in [0,1] \), then \( \exists \{d_n\} \) s.t. \( \liminf_{n \to \infty} m(\{x \mid |f_n(x)| > d_n\}) = 0 \).

4. For each \( n \), \( \exists N_n \) such that \( m(\{x \mid |f_{N_n}(x)| > d_n\}) < \varepsilon_n \). If \( |f_n(x)| \to \infty \) a.e. \( x \in [0,1] \), the Borel-Cantelli lemma yields: \( \exists \{d_n\} \) s.t. \( m(\{x \mid |f_n(x)| > d_n\}) < \varepsilon_n \). Consequently, \( \limsup_{n \to \infty} \sum_{n=1}^{\infty} m(\{x \mid |f_n(x)| > d_n\}) = 0 \) and \( m([0,1] \setminus E) = 0 \).

3. pg 83 Problem 1: Given \( F_1, \ldots, F_n \), \( F_{n+1} \), \( F_1^*, \ldots, F_n^* \), \( F_{n+1}^* \) with \( \mathbb{N} = 2^n - 1 \), \( \bigcup F_j = \bigcup F_j^* \), \( F_j^* \cap F_k^* = 0 \) \( j \neq k \) and \( F_{\infty} = \bigcup F_j^* \). By induction on \( n \), \( n = 1 \) is obvious. For \( n = 2 \), \( N \leq 2^2 - 1 = 3 \). If \( F_1 \cap F_2 = \emptyset \), \( F_j^* = F_j, j = 1,2 \). Otherwise, \( F_1 = F_1 \setminus (F_1 \cap F_2) \), \( F_2 = F_1 \cap F_2 \), \( F_3 = F_2 \setminus F_1 \). Assume the result for \( n \) and consider \( n+1 \). Consider \( \{F_1^*, \ldots, F_n^*, F_{n+1}\} \) with \( N(N) = 2^n - 1 \). For each \( \{F_j^*, \ldots, F_{n+1}\} \setminus \{F_j, \ldots, F_{n+1}\} \neq \emptyset \), \( 2 \) new sets are created, so we get...
At most \(2(2^n - 1) + 1\) (plus 1 from the remaining piece of \(F_{n+1}\))
so \(2^{n+1} - 1\) new sets \(F_1^*, \ldots, F_m^*\). Each \(F_j^*\) satisfies
\[
F_j^* \cap F_k^* = \emptyset \quad \text{for all } j \neq k.
\]
(Use the hint)

Prop 1.1 (ii) \(\phi = \sum a_j \chi_{F_j^*}\) simple then \(\int a \phi + \psi = a \int \phi + \int \psi\), \(a \in \mathbb{R}\). (Proof: Consider the collection \(\{F_j, G_j\}\) (finite)
By the exercise \(\exists \{S_m\}\) disjoint finite family of sets s.t.
\[
U S_m = UF_j \cup UBG_l \quad \text{and} \quad (a \phi + \chi)_{S_m} = a \chi_{S_m}
\]
where
\(a \chi_{S_m}\) is a linear combination of \(a \chi_{F_j}\) and \(a \chi_{G_l}\).
Then \(a \phi + \psi = \sum a \chi_{S_m}\) is simple and
\[
\int a \phi + \psi = \sum a \chi_{S_m} m(S_m) = a \sum a \chi_{S_m} m(S_m)
\]

Since each \(F_j\) or \(G_l\) is the union of \(S_m\)s in it, \(m F_j = \sum m S_m\)
and \(m G_l = \sum m S_m\), we group the sum to get:
\[
\frac{\sum a \chi_{S_m} m(S_m)}{\sum m S_m} = \frac{\sum a \chi_{F_j} m(S_m)}{\sum m S_m}\]

(iv) \(\phi \leq \chi \Rightarrow 0 \leq \chi - \phi\). Since \(\chi - \phi\) is simple (used above to show this) and the integral is linear on simple functions,
\[
\int (\chi - \phi) = \int \chi - \int \phi \geq 0 \quad \text{so} \quad \int \chi \geq \int \phi.
\]

4. (2py. 71 #9) Chebyshev Ineq: \(f \geq 0\), \(\int f = 1\) then \(m E_d \leq \frac{1}{\alpha} \int f\).
pf. \(\alpha \chi_{E_d} \leq f\) since \(E_d = \{x \mid f(x) > \alpha f\}\)
so by \(\chi_{E_d}\) monotonicity of the integral
\[
\int \chi_{E_d} = \alpha m(E_d) \leq \int f < \infty
\]
\[
\Rightarrow m(E_d) \leq \frac{1}{\alpha} \int f.
\]
5. \( f \) odd \( m \) and \( \text{supp } f = E \) has finite \( m \). Then \( \int f \) as defined in 2. equals \( \sum \phi \) and \( \text{inf } \psi \). \( \text{Pf.} \) Take \( t > 0 \). By monotonicity, \( \psi \leq t \Rightarrow \int \psi \leq \int t \). On the other hand, if \( \text{seq } \phi_n \geq 0 \) of simple \( f \)s. \( \phi_n \uparrow f \text{ ptw a.e. so } \lim \int \phi_n = \int f \) (MCT).

Hence \( \sup \int \phi \geq \int f \Rightarrow \int f = \sup \int \phi \). Similarly, if \( \psi \geq f \), monotonicity implies \( \int \psi \geq \int f \). Now as on pg 31, if \( f(x) \leq M \) on \( E = \text{supp } f \), \( m < \infty \), set \( E_m = \{ x \mid \frac{M}{m} \leq f \leq \frac{M+1}{m} \} \)

Then \( \bigcup_{n=0}^{m} E_{n} = E \) and \( \phi = \sum_{n=0}^{m} \frac{m+1}{m} \chi_{E_{n}} \geq f \) and simple.

\( \lim_{m \to \infty} \phi_m = f \) ptw a.e. Since \( \psi \leq N \chi_{E} \text{ LDC} \Rightarrow \sup \int \psi = \int f \)

\( \Rightarrow \text{inf } \int \psi \leq \int f \) or \( \int f = \text{inf } \int \psi \). Note we used \( m < \infty \)

and odd of \( f \). For the general result, \( f = f^{+} - f^{-} \) and work with \( f^{\pm} \) separately.

6. \( f \) odd, \( \text{supp } f = E \), \( m < \infty \). If \( \sup \int \phi = \inf \int \psi \) then \( \phi \leq f \) \( \psi \geq f \).

Pf. As in the proof of 15), since \( |f| \leq M \), we define

\[ E_{n} = \{ x \mid \frac{kM}{n} \leq f(x) \leq \frac{(k+1)M}{n} \} \]

\[ k = -n, \ldots, n \]

Define simple functions \( \psi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} \chi_{E_{k}}(x) \)

and \( \phi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} (k-1) \chi_{E_{k}}(x) \).
so $\varphi_n \leq f \leq \psi_n$ and $\lim_{n \to \infty} \sum_{E \in \mathcal{E}} m(E) = m(E) < \infty$. Since the seq of $\mathcal{E}$, $\varphi_n$ & $\psi_n$ are odd above, resp., below, we have

$\varphi = \sup \{ \varphi_n \leq f \leq \psi_n : \varphi_n = \varphi \text{ are in } \mathcal{E} \}$. Finally, we show that $f = \varphi \leq \psi$ except on a set of $m$ zero so by problem 5 $f$ is measurable. So let $Q = \{ \varphi < 4 \}$ and $Q_n = \{ \varphi < 4 - \frac{1}{m} \}$ so $Q = \bigcup Q_n$. Now $Q_n \subset \{ \varphi_n \leq 4 - \frac{1}{m} \}$ and the $m(\int_{Q_n} \varphi_n < 4 - \frac{1}{m}) \to 0$ as $n \to \infty$, so $m(Q_n) = 0 \Rightarrow m(Q) = 0$. \[\square\]