MA676 : Solutions to PS 4

Spring 09

4-1

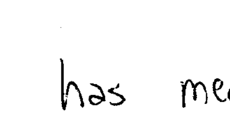
2 1. Property 6: If f m & $f=g$ a.e. then g m. <u>Proof</u> let $E \subset \mathbb{R}^d$ be the set for which $f(x) \neq g(x)$. By hypothesis, $mE=0$. For each $a \in \mathbb{R}$ $\exists$ $E_a \subset E$, $m E_a = 0$, so $\{x \mid g(x) < a\} = (\{x \mid f<a\} - E) \cup E_a$, the union of 2 measurable sets, so it is measurable. ■ $(E_a = \{x \mid g(x) \neq f(x) \text{ and } g(x) < a\}.)$

2. let $\{f_n\}$ be a seq of m fncs on $[0,1]$ with $|f_n(x)| < \infty$ a.e. $x \in [0,1]$. $\exists$ $\{c_n\}$, $c_n > 0$ s.t. $f_n(x)/c_n \to 0$ a.e. x.

<u>Pf</u> For each n $\exists$ d_n such that $m(\{x \mid |f_n(x)| > d_n\}) < 1/2^n$. If not $\exists$ seq. $f_m \nearrow \infty$ s.t. $m(\{x \mid |f_n(x)| > f_m\}) > 1/2^n$ $\forall m$ contradicting the fact that $|f_n(x)| < \infty$ a.e. $x \in [0,1]$.
(By monotonicity of the measure we can assume $d_n \nearrow \infty$)
Set $c_n = n d_n$ so $m(\{x \mid |f_n(x)|/c_n > 1/n\}) < 1/2^n$.
To apply the Borel-Cantelli lemma, set $E_n = \{x \mid |f_n(x)|/c_n > \frac{1}{n}\}$, so $\sum_{n=1}^{\infty} m(E_n) \le \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty$. Then $E = \limsup_n E_n$

has measure zero: $E = \{x \mid \frac{|f_n(x)|}{c_n} > \frac{1}{n}\}$. Consequently, $x \in [0,1] \setminus E$, $\frac{|f_n(x)|}{c_n} < \frac{1}{n} \to 0$ & $m([0,1]\setminus E) = 0$ ■

3. pg 89 Problem 1: Given $F_1, \dots, F_n \subset \mathbb{R}^n$ $\exists$ $F_1^*, \dots, F_N^* \subset \mathbb{R}^n$ with $N \le 2^n - 1$ so $\bigcup F_j = \bigcup F_j^*$, $F_j^* \cap F_k^* = 0$ $j \neq k$ and $F_k = \bigcup_{F_j^* \subset F_k} F_j^*$ $\forall k = 1, \dots, N$. <u>Pf</u> By induction on n. $n=1$ is obvious. For $n=2$, $N \le 2^2 - 1 = 3$. If $F_1 \cap F_2 = \phi$, $F_j^* = F_j$, $j=1,2$.
Otherwise $F_1^* = F_1 \setminus (F_1 \cap F_2)$ $F_2^* = F_1 \cap F_2$, $F_3^* = F_2 \setminus F_1 \cap F_2$

F_1  F_2 $\quad F_1 = F_1^* \cup F_2^*$ $\quad F_2 = F_2^* \cup F_3^*$

Assume the result for n & consider $n+1$. Consider $\{F_1^*, \dots, F_{N(n)}^*, F_{n+1}\}$ w/ $N(n) \le 2^n - 1$. For each set F_j^* s.t. $F_j^* \cap F_{n+1} \neq \phi$, 2 new sets are created so we get

at most $2(2^n-1)+1$ (plus 1 from the remaining piece of F_{n+1})
so $2^{n+1}-1$ new sets $\tilde{F}_1^*, \dots, \tilde{F}^*_{N(n+1)}$. Each $\tilde{F}_j^*$ satisfies
$F_\ell = \bigcup_{F_j^* \subset F_\ell} F_j^* = \bigcup_{F_j^* \subset F_j} \left(\bigcup_{\hat{F}_m^* \subset F_j^*} \tilde{F}_m^* \right) = \bigcup_{F_m^* \subset F_\ell} F_m^*$. ∎ (or use the hint)

Prop 1.1 (ii) $\varphi = \sum c_k \chi_{F_k}$ $\psi = \sum a_\ell \chi_{G_\ell}$ simple then $\int a\varphi + \psi = a\int \varphi + \int \psi$, $a \in \mathbb{R}$. Pf Consider the collection $\{F_j, G_\ell\}$ (finite)
By the exercise $\exists \{S_m\}$ disjoint finite family of sets s.t.
$\bigcup S_m = \bigcup F_j \cup \bigcup G_\ell$ and $(a\varphi + \chi)|_{S_m} = A_m \chi_{S_m}$ where
A_m is a linear combination of ac_k and a_ℓ.
Then $(a\varphi + \psi) = \sum A_m \chi_{S_m}$ is simple and

$$\int a\varphi + \psi = \sum A_m m(S_m), \quad A_m = \begin{cases} a_{j_1} + a_{j_2} + \dots a_{j_k} \text{ if } S_m \subset G_j \\ + c_{k_1} + c_{k_2} + \dots + c_{k_\ell} \end{cases}$$

Since each F_j or G_ℓ is the union of S_m's in it, $mF_j = \sum_{S_m \subset F_j} m S_m$ and $m G_\ell = \sum_{S_m \subset G_\ell} m S_m$, we group the sum to get:

$$\sum_\ell \left(\sum_{S_m \subset G_\ell} a_\ell \, m(S_m) \right) + \sum_j \left(\sum_{S_m \subset F_j} a c_j \, m(S_m) \right) = \left(\sum a_\ell \, m(G_\ell) \right) + a \left(\sum c_j \, m(F_j) \right)$$
$$= \int \psi + a \int \varphi.$$

(iv) $\varphi \le \chi \Rightarrow 0 \le \chi - \varphi$. Since $\chi - \varphi$ is simple (use the above to show this) and the integral is linear on simple functions,
$\int \chi - \varphi = \int \chi - \int \varphi \geq 0$ so $\int \chi \geq \int \varphi$.

4. (S^2 pg 91 #9) Chebyshev Ineq: $f \geq 0$, $f \in L^1$ then $mE_\alpha \le \frac{1}{\alpha} \int f$.
Pf $\alpha \chi_{E_\alpha} \le f$ since $E_\alpha = \{x \mid f(x) > \alpha\}$
so by monotonicity of the integral

$$\alpha \int \chi_{E_\alpha} = \alpha \, m(E_\alpha) \le \int f < \infty$$
$$\Rightarrow m(E_\alpha) \le \frac{1}{\alpha} \int f.$$

5. f bdd m and supp $f = E$ has finite m. Then $\int f$ as defined in §2 equals $\sup_{\varphi \le f} \int \varphi$ and $\inf_{\psi \ge f} \int \psi$. Pf. Take $f \ge 0$. By monotonicity, $\varphi \le f \Rightarrow \sup_{\varphi \le f} \int \varphi \le \int f$. On the other hand, $\exists$ seq $\varphi_n \ge 0$ of simple fncs. $\varphi_n \uparrow f$ ptw a.e. so $\lim \int \varphi_n = \int f$ (MCT)

hence $\sup_{\varphi \le f} \int \varphi \ge \int f \Rightarrow \int f = \sup_{0 \le \varphi \le f} \int \varphi$. Similarly, if $\psi \ge f$, monotonicity implies $\inf_{\psi \ge f} \int \psi \ge \int f$. Now as on pg 31, if $f(x) \le N$ on $E = \text{supp} f$, $mE < \infty$, set $E_{\ell,M} = \{x \mid \frac{\ell}{M} \le f \le \frac{\ell+1}{M}\}$ then $\bigcup_{\ell=0}^{NM} E_{\ell,M} \supseteq E$ and $\varphi_M = \sum_{\ell=0}^{NM} \frac{\ell+1}{M} \chi_{E_{\ell,M}} \ge f$ and simple. $\lim_{M\to\infty} \varphi_M = f$ ptw a.e. Since $\varphi_M \le N\chi_E$ LDC $\Rightarrow \lim \int \varphi_M = \int f$

$\Rightarrow$ $\inf_{\varphi \ge f} \int \varphi \le \int f$ or $\int f = \inf_{\psi \ge f} \int \psi$. Note we used $mE < \infty$ and bdd of f. For the general result, $f = f^+ - f^-$ and work with $f^\pm$ separately ∎.

6. f bdd, supp $f = E$ m, $mE < \infty$. If $\sup_{\varphi \le f} \int \varphi = \inf_{\psi \ge f} \int \psi$ then f is m.

Pf As in the proof of (5), since $|f| \le M$, we define
$$E_k = \left\{x \mid \frac{kM}{n} \ge f(x) > \frac{(k-1)M}{n}\right\} \quad k = -n, \dots, n.$$

Define simple functions $\psi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} k \chi_{E_k}(x)$ and
$$\varphi_n(x) = \frac{M}{n} \sum_{-n}^{n} (k-1) \chi_{E_k}(x)$$

so $\varphi_n \le f \le \psi_n$ and $\lim_{n\to\infty} \sum_{k=-n}^{n} mE_k = mE < \infty$. Since the seq of m fncs φ_n & ψ_n are bdd above, resp., below, we have

$\varphi = \sup_n \varphi_n \le f \le \inf_n \psi_n = \psi$ are m fncs. Finally, we show that $f = \psi = \varphi$ except on a set of m zero so by problem 5 f is measurable. So let $Q = \{\varphi < \psi\}$ and $Q_m = \{\varphi < \psi - \frac{1}{m}\}$ so $Q = \bigcup_m Q_m$. Now $Q_m \subset \{x \mid \varphi_n < \psi_n - \frac{1}{m}\}$ and the $m(\{x \mid \varphi_n < \psi_n - \frac{1}{m}\}) \to 0$ as $n \to 0$ $\Rightarrow$ $mQ_m = 0$ $\forall m$ $\Rightarrow$ $mQ = 0$. ∎