MA 676: Solution to PS #5

1. $f \geq 0$, $f \in L$, & $E_a = \{f > a\}$ then $\lim a\, m E_a = 0$.

<u>Pf</u> let $f_n = f \chi_{E_n}$. Since $f \in L^1$, f is finite a.e. so $\lim_{n\to\infty} f_n = 0$ a.e. since $0 \leq f_n \leq f \in L^1$ LDC $\Rightarrow \lim \int f_n = 0$.

Then: $0 \leq n \cdot m(E_n) \leq \int_{E_n} f \to 0 \Rightarrow \lim_{n\to\infty} n(mE_n) = 0$. ∎

2. f, g m on $\mathbb{R}^d$ then $f(x)g(y)$ m on $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$.

<u>Pf</u> Start with $E_1, E_2 \subset \mathbb{R}^d$ m. Then $E_1 \times E_2 \subset \mathbb{R}^{2d}$ is m so $\chi_{E_1}(x)\chi_{E_2}(y)$, is m (see the text). Extend to simple functions $\varphi = \sum c_j \chi_{F_j}$ & $\psi = \sum b_\ell \chi_{E_\ell}$ where $F_j, E_\ell \subset \mathbb{R}^d$ are m:

$$\varphi(x)\psi(y) = \sum_{j,\ell} b_\ell c_j \chi_{F_j}(x)\chi_{E_\ell}(y)$$

and since each $\chi_{F_j}(x)\chi_{E_\ell}(y)$ is $\mathbb{R}^{2d}$-m, so is $\varphi.\psi(x,y)$. Finally, if f & g are $\mathbb{R}^d$-m, $\exists$ seq simple fncs. $\varphi_k \to f$ & $\psi_k \to g$ a.e. $\Rightarrow \varphi_k(x)\psi_k(y) \to f(x)g(y)$ a.e. $x, y \in \mathbb{R}^d$. Since $\varphi_k(x)\psi_k(y)$ is a simple fnc on $\mathbb{R}^{2d}$, note that $\chi_E(x)\chi_F(y) = \chi_{E\times F}(x,y)$ as we used above, we get that $f(x)g(y)$ is the pointwise a.e. limit of a seq of m fncs and hence m on $\mathbb{R}^{2d}$. ∎ (Note: conv. is a.e. on $\mathbb{R}^{2d}$ so if $\varphi_k \to f$ a.e. ($Z_f \subset \mathbb{R}^d$ m $Z_f = 0$) & same for $\psi_k \to g$ (Z_g) then $\varphi_k(x)\psi_k(y) \to f(x)g(y)$ on $\mathbb{R}^{2d} \setminus (Z_f \times Z_g)$ and $m(Z_f \times Z_g) = 0$.)

(5-2)

3. $R(f,E) \equiv$ region under f over $E \subset \mathbb{R}^{d+1}$ & $\Gamma(f,E) = \{(x, f(x)) \subset \mathbb{R}^{d+1} \mid x \in E,\ f(x) < \infty\}$. Recall $\{x \in \mathbb{R}^d \mid f(x) = +\infty\}$ has d-m zero.

<u>Part 1</u> It is important to note that $m_{d+1}(\Gamma(f,E)) = 0$ (the graph has 0 $d+1$-m)

We proved in S^2 that if $E_j \subset \mathbb{R}^{d_j}$ m then $E_1 \times E_2 \subset \mathbb{R}^{d_1+d_2}$ is m (pg 83). This

So if $E \subset \mathbb{R}^d$ m, $\chi_E \geq 0$ and $R(\chi_E, \mathbb{R}^d)$ is m in $\mathbb{R}^{d+1}$.

If $f = \sum c_j \chi_{F_j}$ is simple then one sees that $R(f, \mathbb{R}^d)$ is m.

Finally $f \geq 0$ m $\exists$ seq simple fnc $f_k \uparrow f$ and $\lim_k R(f_k, \mathbb{R}^d) \cup \Gamma(f, \mathbb{R}^d) = R(f, \mathbb{R}^d)$, so $R(f, \mathbb{R}^d)$ is m. (one proves $m\,\Gamma(f,E) = 0$)

$$\lim_{k\to\infty} \int f_k = \int f = \lim_k \int_{R(f_k, \mathbb{R}^d)} = m_{d+1} R(f, \mathbb{R}^d)$$

To apply Tonelli's Th., note that $\chi_{R(f,E)} \geq 0$ (and integrable, although not needed) and the y-slice $\chi^y_{R(f,E)}(x) \in L^1$ a.e. y. Then

$$\int f = \int \chi_{R(f,\mathbb{R}^d)} = \int_0^\infty dy \left(\int \chi^y_{R(f,\mathbb{R}^d)}(x)\, dx \right)$$

($y \in [0,\infty)$ since $f \geq 0$) For fixed y $\{x \mid f(x) \geq y\} = E_y$ so $\int \chi^y_{R(f,\mathbb{R}^d)}(x)\,dx = m E_y$ so $\int f = \int_0^\infty dy\, m E_y$. ∎

4. Egorov's Th $\Rightarrow$ BCT. <u>Pf</u> $f_n \to f$ ptw a.e. on E, $mE < \infty$ and $|f_n| \le M$ on E. By Egorov for $\varepsilon > 0$ $\exists$ $A_\varepsilon \subset E$, $m(E - A_\varepsilon) \le \varepsilon$ s.t. $f_n \to f$ unif. on A_ε. Then $\exists$

$$\int_E |f-f_n| = \int_{E \setminus A_\varepsilon} |f-f_n| + \int_{A_\varepsilon} |f-f_n|$$

$$\le \|(f-f_n)\chi_{A_\varepsilon}\|_\infty (mE) + 2M\varepsilon$$

By uniform conv, given $\varepsilon > 0$ $\exists$ N_ε s.t. $\|(f-f_n)\chi_{A_\varepsilon}\|_\infty < \varepsilon$ for $n > N_\varepsilon$ so

$$\int_E |f-f_n| < \varepsilon[(mE) + 2M]$$

$$\Rightarrow \lim_{n\to\infty} \int_E |f-f_n| = 0. \blacksquare$$

5. If $f \in L^1([0,1])$, $x^k f \in L^1([0,1])$ and $\lim \int x^k f = 0$.
<u>Pf</u> $|x^k f| < |f|$ since $x \in [0,1]$ and
$\lim_{k\to\infty} x^k f = 0$ pt. a.e. on $[0,1]$ since f is finite a.e.

By LDC $\int_0^1 x^k f \to 0$ $\blacksquare$