MA676 Spring 2009
Hour Test - 100 Points
11 March 2009

INSTRUCTIONS: PLEASE WORK ALL THE PROBLEMS BELOW. EACH PROBLEM IS WORTH 25 POINTS. NO BOOKS, PAPERS, OR NOTES ARE ALLOWED.

NAME: Solutions

Problem 1.

i. Give the definition of a Lebesgue measurable set.

ii. Prove the following theorem

> **Theorem** *A subset $E \subset R^n$ is measurable if and only if $E = H - Z$, where H is a G_δ-type set and Z has measure zero.*

(i) $E \subset \mathbb{R}$ is m if $\forall \varepsilon > 0$ $\exists\, \mathcal{O}_\varepsilon$ open, $E \subset \mathcal{O}_\varepsilon$ s.t.
5 $m_*(\mathcal{O}_\varepsilon - E) \le \varepsilon$. ($m_* \equiv$ outer measure).

(ii) $\Rightarrow$ $E \subset \mathbb{R}^d$ m then $\forall n \in \mathbb{N}$ $\exists\, \mathcal{O}_n$ open, $E \subset \mathcal{O}_n$
15 and $m_*(\mathcal{O}_n - E) \le \frac{1}{n}$. Set $H = \bigcap_{n=1}^{\infty} \mathcal{O}_n \subset \mathcal{O}_m\ \forall m$, $E \subset H$.
H is a G_δ-set. Since $H - E \subset \mathcal{O}_m - E\ \forall m$,

$$m(H - E) \le m(\mathcal{O}_m - E) \le \frac{1}{m}\ \forall m$$

(we can use m or m_* here)

so $Z \equiv H - E$ has m zero. Since $H = E \cup Z$, $E \cap Z = \emptyset$,
$E = H - Z$.

5 $\Leftarrow$ If $E = H - Z$, E is measurable since both H (a Borel set) and Z (a set of m zero) are measurable. ■

Problem 2.

i. Give two conditions for the measurability of a function $f : E \subset R^n \to R$, with E measurable, and prove that they are equivalent.

ii. If $\{f_k\}$ is a sequence of measurable functions on a measurable subset $E \subset R^n$, then the function $\limsup_k f_k$ is a measurable function on E.

(10)

(i) Basic defn: $f : E \subset \mathbb{R}^n \to \mathbb{R}$, E measurable, if $\{x \in E \mid f(x) < a\}$, $\forall a \in \mathbb{R}$, is a measurable set. Equivalent forms

- $\{x \in E \mid f(x) \leq a\}$ m $\forall a \in \mathbb{R}$;
- $\{x \in E \mid f(x) > a\}$ m $\forall a \in \mathbb{R}$
- $\{x \in E \mid f(x) \geq a\}$ m $\forall a \in \mathbb{R}$.

proofs are given on page 28.

(15)

(ii) By defn., $\limsup_k f_k = \inf_k \{ \sup_{n \geq k} f_n \}$.

First, $\sup_{n \geq k} f_n$ is a m fnc. since $\forall a \in \mathbb{R}$: $\{x \mid \sup_{n \geq k} f_n(x) > a\}$

$= \bigcup_{n \geq k} \{x \mid f_n(x) > a\}$ and as each set

$\{x \mid f_n(x) > a\}$ is m & a countable union of m sets is m,

we conclude $\underbrace{\{ \sup_{n \geq k} f_n > a \}}_{E_k^{(a)}}$ is m for each $k \in \mathbb{N}$.

Second, As $\inf_k E_k^{(a)}$ is m, we obtain the measurability of $\limsup f_k$.

Problem 3. Suppose that $E_k \subset R^n$ is a sequence of measurable sets with $m(E_k) < \infty$ and increasing monotonically (that is $E_k \subset E_{k+1}$) to a set E. Prove that E is measurable and that

$$\lim_{k\to\infty} m(E_k) = m(E).$$

$E = \bigcup_k E_k$ is measurable since a count. union of m sets is m.

Set $F_k = E_k - E_{k-1}$ $k = 2, 3, \ldots$ and $F_1 = E_1$

Since $E_k \subset E_{k+1}$, the sets F_k are disjoint

$F_k \cap F_j = \phi$, $k \neq j$, and each F_k is measurable.

Note that $E = \bigcup_{j=1}^{\infty} F_j$ so by countable additivity

$$mE = \sum_{j=1}^{\infty} mF_j$$

Since $\bigcup_{j=1}^{N} F_j = E_N$ $(= E_1 \cup (E_2 - E_1) \cup (E_3 - E_2) \cup \cdots \cup (E_N - E_{N-1}))$

and $\sum_{j=1}^{N} mF_j = mE_N$ we have

$$\lim_{N\to\infty} mE_N = mE$$

(limit of the seq of partial sums).

Problem 4.

(i) If $f \geq 0$ and measurable, define $\int f$.

(ii) State Fatou's Lemma.

(iii) Prove that if $\{f_k\}$ is a sequence of nonnegative measurable functions such that $f_k \to f$ a. e. on R^d, and $0 \leq f_k \leq f$, then $\int f = \lim_{k\to\infty} \int f_k$.

(i) $\int f = \sup\limits_{0 \leq \varphi \leq f} \int \varphi$, where φ is a nonneg m. function
5 with $m(\operatorname{supp} \varphi) < \infty$.

(ii) Fatou's lemma: Let $f_n \geq 0$ be a seq. of nonnegative
5 measurable fncs. s.t. $f_k \to f$ a.e. Then f is m and

$$0 \leq \int f \leq \liminf_k \int f_k \quad (1)$$

(iii) Proof By Fatou's lemma, inequality (1) holds.
15 Now by $0 \leq f_n \leq f$, monotonicity of the integral means

$$0 \leq \int f_n \leq \int f$$

so

$$0 \leq \limsup_n \int f_n \leq \int f \leq \liminf_n \int f_k$$

Since the $\liminf \int f_n \leq \limsup \int f_n$ we have:

$$\limsup \int f_n = \liminf \int f_n = \int f = \lim \int f_k. \blacksquare$$