

**MA677 Fall 2009**  
**Homework Problem Set #2**  
**October 8, 2009**

Assignment 2. Finish reading chapter 4 of Stein-Shakarchi on Hilbert space theory and section 2 of chapter 3 on Good Kernels. You might also look at Wheeden and Zygmund, chapter 8 ( $L^p$ -theory) and chapter 9 (approximations of the identity). These problems are due Friday, 16 October 2009. ( WZ means the problems are from Wheeden-Zygmund).

- (1) (WZ, problem 11, page 144). If  $f_k \rightarrow f$  in  $L^p$ ,  $1 \leq p < \infty$ , and  $g_k \rightarrow g$  pointwise with  $\|g_k\|_\infty < M < \infty$  for all  $k$ , then prove that  $f_k g_k \rightarrow fg$  in  $L^p$ .
- (2) Prove that  $C_0^\infty(\mathbb{R}^d)$  is dense in  $L^p(\mathbb{R}^d)$ , for  $1 \leq p < \infty$ . Use convolutions and a good choice of an approximation of the identity.
- (3) (WZ, problem 12, page 144). Let  $f$  and  $f_k$  be  $L^p$  functions. Prove that if  $f_k \rightarrow f$  in  $L^p$ , then  $\|f_k\|_p \rightarrow \|f\|_p$ . Conversely, prove that if  $f_k \rightarrow f$  pointwise a. e. and if  $\|f_k\|_p \rightarrow \|f\|_p$ , then  $f_k \rightarrow f$  in  $L^p$ .
- (4) (WZ, problem 15, page 144.) Assume that  $\cos kx$  and  $\sin kx$ , for  $k \in \mathbb{Z}$ , form an orthogonal basis for  $L^2(0, 2\pi)$ . Actually, we proved this in class. Why? Let  $\psi_k(x)$  be the resulting ONB. Prove that if  $f \in L^1(0, 2\pi)$ , then  $(f, \psi_k) \rightarrow 0$  as  $k \rightarrow \infty$ . To prove this, first assume  $f \in L^2(0, 2\pi)$ , and then approximate an  $L^1$  function  $f$  as  $f = g + h$ , with  $g \in L^2$  and  $\|h\|_1$  arbitrarily small.
- (5) Exercise 18, page 197, of Stein-Shakarchi.