Assignment 2. Finish reading chapter 4 of Stein-Shakarchi on Hilbert space theory and section 2 of chapter 3 on Good Kernels. You might also look at Wheeden and Zygmund, chapter 8 ($L^p$-theory) and chapter 9 (approximations of the identity). These problems are due Friday, 16 October 2009. (WZ means the problems are from Wheeden-Zygmund).

1. (WZ, problem 11, page 144). If $f_k \to f$ in $L^p$, $1 \leq p < \infty$, and $g_k \to g$ pointwise with $\|g_k\|_\infty < M < \infty$ for all $k$, then prove that $f_kg_k \to fg$ in $L^p$.

2. Prove that $C_0^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, for $1 \leq p < \infty$. Use convolutions and a good choice of an approximation of the identity.

3. (WZ, problem 12, page 144). Let $f$ and $f_k$ be $L^p$ functions. Prove that if $f_k \to f$ in $L^p$, then $\|f_k\|_p \to \|f\|_p$. Conversely, prove that if $f_k \to f$ pointwise a.e. and if $\|f_k\|_p \to \|f\|_p$, then $f_k \to f$ in $L^p$.

4. (WZ, problem 15, page 144.) Assume that $\cos kx$ and $\sin kx$, for $k \in \mathbb{Z}$, form an orthogonal basis for $L^2(0,2\pi)$. Actually, we proved this in class. Why? Let $\psi_k(x)$ be the resulting ONB. Prove that if $f \in L^1(0,2\pi)$, then $(f, \psi_k) \to 0$ as $k \to \infty$. To prove this, first assume $f \in L^2(0,2\pi)$, and then approximate an $L^1$ function $f$ as $f = g + h$, with $g \in L^2$ and $\|h\|_1$ arbitrarily small.

5. Exercise 18, page 197, of Stein-Shakarchi.