MA681–001 Functional Analysis
Fall 2013
Problem Set 3
DUE: Monday, 14 October 2013

1. Write the argument in the proof of Theorem III.10 in Reed and Simon that reduces the problem to showing that $T[B^X(0,r)]$ has nonempty interior for some $r > 0$.

2. Prove: A normed linear vector space is complete if and only if every absolutely summable sequence is summable. Note that a sequence $\{x_j\} \subset X$ in a NLVS X is absolutely summable if $\sum_{j=1}^{\infty} \|x_j\| < \infty$, and it is summable if the sequence of partial sums $\sum_{j=1}^{N} x_j$ has a limit in X.

3. Let V be an inner product space and let $\{x_j\}_{j=1}^{N}$ be an orthonormal set: $\|x_j\| = 1$ and $\langle x_i, x_j \rangle = 0, i \neq j$. Prove that for any $x \in V$, the quantity

$$\left\| x - \sum_{j=1}^{N} c_j x_j \right\|$$

is minimized with the choice of $c_j = \langle x_j, x \rangle$. This is the basis of the least squares method.

4. Let $\mathcal{H}$ be a Hilbert space.

   (a) Prove that a strongly convergent sequence converges weakly but a weakly convergent sequence need not converge strongly.

   (b) Prove that if a sequence $\{x_j\}$ converges strongly, than the sequence $\|x_j\|$ converges.