MA681–001 Functional Analysis
Fall 2016
Problem Set 1
DUE: Wednesday, 7 September 2016

1. Show that $C([0, 1])$ with the sup-norm is a Banach space. Let $A$ be the set of all polynomials with rational complex coefficients (that is, those of the form $a + ib$ with $a, b \in Q$). Show that $A$ is dense in $C([0, 1])$. Conclude that this Banach space is separable.

2. Banach Algebras. A Banach algebra is a Banach space that is also an algebra. This means that there is a multiplication of elements which is compatible with all the other structures. In particular, if $x, y \in A$, then $xy \in A$ and $\|xy\| \leq \|x\| \|y\|$, and the multiplication is distributive. Show that $M_n(C)$ and $(C([0, 1]), \| \cdot \|_{\infty})$ are Banach algebras.

3. Suppose $(X, \langle \cdot, \cdot \rangle)$ is an inner product space. Show that there is a Hilbert space $(\tilde{X}, \langle \cdot, \cdot \rangle_{\tilde{X}})$ so that $X$ is isomorphic with a dense subset of $\tilde{X}$ and the inner product satisfies $\langle \tilde{x}, \tilde{y} \rangle_{\tilde{X}} = \langle x, y \rangle_X$ for all $x, y \in X$. You may use relevant parts of the discussion in class.


5. Postponed to a later PS. Consider the Banach space $(C([0, 1]), \| \cdot \|_{\infty})$. Let $a \in C([0, 1])$ and define the multiplication operator $A : f \in C([0, 1]) \rightarrow af$. Show that $A$ is a bounded linear operator and compute its norm. If $|a| < 1$, show that the operator $1 + A$ is a boundedly invertible operator.