MA681–001 Functional Analysis
Fall 2016
Problem Set 1
DUE: Wednesday, 7 September 2016

1. Show that $C([0,1])$ with the sup-norm is a Banach space. Let A be the set of all polynomials with rational complex coefficients (that is, those of the form $a+ib$ with $a,b \in Q$). Show that A is dense in $C([0,1])$. Conclude that this Banach space is separable.

2. Banach Algebras. A *Banach algebra* is a Banach space that is also an algebra. This means that there is a multiplication of elements which is compatible with all the other structures. In particular, if $x,y \in A$, then $xy \in A$ and $\|xy\| \leq \|x\| \, \|y\|$, and the multiplication is distributive. Show that $M_n(C)$ and $(C([0,1]), \|\cdot\|_\infty)$ are Banach algebras.

3. Suppose $(X, \langle \cdot, \cdot \rangle)$ is an inner product space. Show that there is a Hilbert space $(\tilde{X}, \langle \cdot, \cdot \rangle_{\tilde{X}})$ so that X is isomorphic with a dense subset of $\tilde{X}$ and the inner product satisfies $\langle \tilde{x}, \tilde{y} \rangle_{\tilde{X}} = \langle x, y \rangle_X$ for all $x, y \in X$. You may use relevant parts of the discussion in class.

4. Conway, page 7, Problem 11.

5. **Postponed to a later PS.** Consider the Banach space $(C([0,1]), \|\cdot\|_\infty)$. Let $a \in C([0,1])$ and define the multiplication operator $A : f \in C([0,1]) \rightarrow af$. Show that A is a bounded linear operator and compute it's norm. If $|a| < 1$, show that the operator $1 + A$ is a boundedly invertible operator.