

MA681–001 Functional Analysis

Fall 2016

Problem Set 5

DUE: Wednesday, 9 November 2016

- (1) Let X be a nontrivial normed linear vector space and fix a nonzero $w \in X$. Then there is a nonzero bounded linear functional Λ on X so that $\Lambda(w) = \|w\|$ and $\|\Lambda\|_{X^*} = 1$.
- (2) Use the Baire Category Theorem to prove that any Hamel basis of a non-finite dimensional Banach space must be uncountable.
- (3) Let X be a Banach space. Prove that a weakly open set is norm open and that a weakly convergent sequence is norm bounded.
- (4) Let X be a linear vector space with a countable family of semi-norms $\{\rho_n\}$. Prove that

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left[\frac{\rho_n(x - y)}{1 + \rho_n(x - y)} \right]$$

is a metric on X .