MA681–001 Functional Analysis
Fall 2016
Problem Set 5
DUE: Wednesday, 9 November 2016

(1) Let $X$ be a nontrivial normed linear vector space and fix a nonzero $w \in X$. Then there is a nonzero bounded linear functional $\Lambda$ on $X$ so that $\Lambda(w) = \|w\|$ and $\|\Lambda\|_{X^*} = 1$.

(2) Use the Baire Category Theorem to prove that any Hamel basis of a non-finite dimensional Banach space must be uncountable.

(3) Let $X$ be a Banach space. Prove that a weakly open set is norm open and that a weakly convergent sequence is norm bounded.

(4) Let $X$ be a linear vector space with a countable family of semi-norms $\{\rho_n\}$. Prove that

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left[ \frac{\rho_n(x - y)}{1 + \rho_n(x - y)} \right]$$

is a metric on $X$. 