

RESONANCES FOR MANIFOLDS HYPERBOLIC NEAR INFINITY: OPTIMAL LOWER BOUNDS ON ORDER OF GROWTH

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ABSTRACT. Suppose that (X, g) is a conformally compact $(n + 1)$ -dimensional manifold that is hyperbolic near infinity in the sense that the sectional curvatures of g are identically equal to minus one outside of a compact set $K \subset X$. We prove that the counting function for the resolvent resonances has maximal order of growth $(n + 1)$ generically for such manifolds. This is achieved by constructing explicit examples of manifolds hyperbolic at infinity for which the resonance counting function obeys optimal lower bounds.

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1. INTRODUCTION

Resonances are poles of the resolvent for the Laplacian on a non-compact manifold. Resonances are the natural analogue of the eigenvalues of the Laplacian on a compact manifold: they are closely related to the classical geodesic flow, and determine asymptotic behavior of solutions of the wave equation.

A fundamental object of interest is the resonance counting function, $N(r)$, defined as the number of resonances (counted with appropriate multiplicity) in a disc

Date: June 24, 2010.

DB supported in part by NSF grant DMS-0901937.

TC supported in part by NSF grants DMS-0500267 and DMS-1001156.

PDH supported in part by NSF grant DMS-0803379.

PP supported in part by NSF grant DMS-0710477.

of radius r about a chosen fixed point in the complex plane. Upper bounds on the resonance counting function of the Laplacian on a Riemannian manifold (X, g) typically take the form $N(r) \leq Cr^m$ for large r , where $m = \dim X$. Lower bounds on the resonance counting function (which imply the existence of the resonances) are typically much harder to prove.

The purpose of this paper is to prove optimal lower bounds on the order of growth of the resonance counting function for generic metrics in a class of manifolds hyperbolic near infinity. Here the *order of growth* of a counting function $N(r)$ is defined to be

$$(1.1) \quad \rho = \limsup_{r \rightarrow \infty} \left(\frac{\log N(r)}{\log r} \right),$$

and we say that the resonance counting function of the Laplacian on a Riemannian (X, g) with $\dim X = m$ has *maximal order of growth* if $\rho = m$. If the resonance counting function does not have maximal order of growth, we will say that (X, g) is *resonance-deficient*. We will prove that, among compactly supported metric perturbations of a given metric g_0 in our class, the set of metrics whose resonance counting function has maximal order of growth is a dense G_δ set or better; the precise formulation is given in Theorem 1.1.

In even dimensions, the nature of the singularity of the wave trace at zero makes it easy to obtain generic lower bounds on the resonance counting function. Hence the main challenge lies in the odd-dimensional case. Our work here draws on two principle sources: first, Sjöstrand and Zworski's [32] construction of an asymptotically Euclidean metric whose resonance counting function obeys a lower bound of the form $N(r) \geq Cr^m$, and second, the techniques developed by Christiansen [4], [6], and Christiansen-Hislop [7] to prove lower bounds on the resonance counting function for generic potentials and metrics.

Sjöstrand and Zworski constructed their example of an asymptotically Euclidean metric with many resonances by gluing a large sphere onto Euclidean space. They exploit the singularity of the wave trace for the Laplacian on the sphere from its periodic geodesics, and show that this singularity persists under gluing. Using the Poisson formula for resonances and a Tauberian argument, they obtain lower bounds on the counting function.

Here we will use elementary propagation estimates for the wave equation together with a Poisson formula due to Borthwick [2] to show that this same gluing construction can be carried out perturbatively on a large class of manifolds with nontrivial geometry and topology. This class consists of conformally compact manifolds with constant curvature -1 in a neighborhood of infinity, described in greater detail in what follows. Then, we will use Christiansen's method to show that, generically within this class, the counting functions have maximal order of growth.

Christiansen's method was developed in the context of Euclidean scattering. It requires that the basic objects of scattering theory (the scattering operator and scattering phase) remain well-behaved under complex perturbations of the potential or metric, and also requires that at least one potential or metric in the class has a resonance counting function with maximal order of growth. Christiansen's method then shows that the same is true for a dense G_δ set of metrics or potentials. Such results are "best possible" in the sense that there are known examples where the resolvent is entire and there are *no* resonances (see [5] and see comments in what follows). One of our contributions here is to provide a robust method for

constructing such examples which relies only on the existence of a “good” Poisson formula for resonances and elementary propagation estimates on the wave operator which hold for any Riemannian manifold.

We now describe the geometric setting for our results in greater detail. Let \bar{X} be a compact manifold with boundary having dimension $m = n + 1$, and denote by X the interior of \bar{X} . Suppose that x is a defining function for the boundary of \bar{X} , that is, a smooth function on \bar{X} with $x > 0$ in X which vanishes to first order on $M = \partial\bar{X}$. Two such defining functions differ at most by a smooth positive function that does not vanish at $\partial\bar{X}$. A complete metric g on X with the property that x^2g extends to a smooth metric on \bar{X} is called *conformally compact*. As x ranges over admissible defining functions, the metrics

$$h_0 = x^2g|_{T^*\partial\bar{X}}$$

give M a natural conformal structure. If $[h_0]$ denotes the conformal class of h_0 , the conformal manifold $(M, [h_0])$ is called the *conformal infinity* of (X, g) . A motivating example is the case where X is the quotient of real hyperbolic $(n + 1)$ -dimensional space by a convex co-compact discrete group of isometries, so that X has infinite metric volume and no cuspidal ends.

A conformally compact manifold (X, g) is called *asymptotically hyperbolic* if the sectional curvatures approach -1 as $x \downarrow 0$, and *hyperbolic near infinity* if the sectional curvatures of g are identically -1 outside a compact subset K of X . Finally, (X, g) is *strongly hyperbolic near infinity* if the following slightly more stringent condition holds: there is a compact subset K of X , a convex co-compact hyperbolic manifold (X_0, g_0) , and a compact subset K_0 of X_0 so that $(X - K, g)$ is isometric to $(X_0 - K_0, g_0)$. We will consider scattering theory and resonances for manifolds hyperbolic near infinity.

We recall some fundamental results in the spectral and scattering theory for asymptotically hyperbolic manifolds. See the papers of Mazzeo-Melrose [27], Joshi-Sa Barreto [22, 24] for spectral and scattering on asymptotically hyperbolic manifolds, see the papers of Guillopé-Zworski [18, 19, 20] for spectral and scattering on manifolds hyperbolic near infinity, and see the papers of Graham-Zworski [12] and Guillarmou [13, 15] for further results on scattering resonances and resolvent resonances. A survey and further references can be found in [30].

If (X, g) is hyperbolic near infinity, the positive Laplacian Δ_g on X has at most finitely many discrete eigenvalues and continuous spectrum in $[n^2/4, \infty)$. The resolvent

$$(1.2) \quad R_g(s) = (\Delta_g - s(n - s))^{-1},$$

initially defined for $\Re(s) > n/2$, extends to a meromorphic family of operators mapping $\mathcal{C}_0^\infty(X)$ into $\mathcal{C}^\infty(X)$. The singularities of the meromorphically continued resolvent (excepting essential singularities) are called *resolvent resonances*. At each resolvent resonance ζ the resolvent has a Laurent series with finite polar part whose coefficients are finite-rank operators. If (X, g) is hyperbolic near infinity, the resolvent has no essential singularities, as the construction in [18] shows. The multiplicity of a resolvent resonance ζ is given by

$$(1.3) \quad m_g(\zeta) = \text{rank } \text{Res}_{s=\zeta} R_g(s).$$

Note that there may be finitely many poles ζ with $\Re(\zeta) > n/2$ corresponding to the finitely many eigenvalues $\lambda = \zeta(n - \zeta)$ of Δ_g . We denote by \mathcal{R}_g the resolvent resonances of Δ_g , counted with multiplicity.

Our interest lies in the asymptotic behavior of the counting function for resolvent resonances:

$$(1.4) \quad N_g(r) = \# \{ \zeta \in \mathcal{R}_g : |\zeta - n/2| \leq r \}.$$

Optimal upper bounds of the form $N_g(r) \leq Cr^{n+1}$ were proven by Cuevas-Vodev [8] and Borthwick [2], but, for reasons that we will explain, optimal lower bounds for resolvent resonances are more difficult to obtain. In the case $n = 1$, Guillopé and Zworski proved sharp upper [19] and lower [20] bounds.

We will study the distribution of resolvent resonances using the Poisson formula for resonances obtained by Guillopé and Zworski for $n = 1$ in [19] and in the present setting by the first author in [2]. To state it, we recall the 0-trace, a regularization introduced by Guillopé and Zworski [19] and inspired by the b -integral of Melrose [28]. First, the 0-integral of a function $f \in \mathcal{C}^\infty(X)$, polyhomogeneous in x as $x \downarrow 0$, is defined to be

$$\int^0 f dg = \text{FP}_{\varepsilon \downarrow 0} \int_{x > \varepsilon} f dg,$$

and for an operator A with smooth kernel we define the 0-trace to be the 0-integral of the kernel of A on the diagonal. The 0-volume of (X, g) , denoted $0\text{-Vol}(X, g)$, is simply $\int^0 dg$ and is known to be independent of the choice of x if the dimension of X is even. In [2], Borthwick proved that if (X, g) is strongly hyperbolic near infinity, then

$$(1.5) \quad 0\text{-Tr} \cos(t\sqrt{\Delta_g - n^2/4}) = \sum_{\zeta \in \mathcal{R}_g^{\text{sc}}} e^{(\zeta - n/2)|t|} - A(X) \frac{\cosh(|t|/2)}{2(\sinh(|t|/2))^{n+1}}$$

where

$$(1.6) \quad A(X) = \begin{cases} 0, & n \text{ odd,} \\ |\chi(\overline{X})|, & n \text{ even} \end{cases}$$

and the left-hand side is a distribution on $\mathbb{R} \setminus \{0\}$, where $\chi(\overline{X})$ is the Euler characteristic of \overline{X} viewed as a compact manifold with boundary. The set $\mathcal{R}_g^{\text{sc}}$ is the set of *scattering resonances* of Δ_g , a set which contains the resolvent resonances but also contains new singularities which arise owing to the conformal infinity. The scattering resonances are singularities of the scattering operator for Δ_g , which we now describe.

Fix a defining function x for $\partial\overline{X}$ and consider the Dirichlet problem for given $s \in \mathbb{C}$ and $f \in \mathcal{C}^\infty(M)$:

$$(1.7) \quad \begin{aligned} (\Delta_g - s(n - s))u &= 0 \\ u &= x^{n-s}F + x^sG \\ F|_{\partial\overline{X}} &= f. \end{aligned}$$

Here, the functions F and G are restrictions to X of smooth functions on \overline{X} . The Dirichlet problem (1.7) has a unique solution if $\Re(s) = n/2$, $s \neq n/2$, so that for

such s the map

$$(1.8) \quad \begin{aligned} S_g(s) &: \mathcal{C}^\infty(\partial\bar{X}) \rightarrow \mathcal{C}^\infty(\partial\bar{X}) \\ f &\mapsto G|_{\partial\bar{X}} \end{aligned}$$

is well-defined and unitary. The scattering operator extends to a meromorphic operator-valued function of s , but with poles whose residues have infinite rank. If we renormalize and set

$$(1.9) \quad \tilde{S}_g(s) = \frac{\Gamma(s - n/2)}{\Gamma(n/2 - s)} S_g(s),$$

the poles with infinite-rank residues are removed and all poles of $\tilde{S}_g(s)$ have finite-rank residues. Poles of $\tilde{S}_g(s)$ are called *scattering resonances*, and the multiplicity of a scattering resonance ζ is given by

$$(1.10) \quad \nu_g(\zeta) = -\operatorname{tr} \operatorname{Res}_{s=\zeta} \left[\tilde{S}'_g(s) \tilde{S}_g(n-s) \right].$$

We denote by $\mathcal{R}_g^{\operatorname{sc}}$ the set of scattering resonances for Δ_g , counted with multiplicity and we denote by $N_g^{\operatorname{sc}}(r)$ the counting function analogous to (1.4):

$$N_g^{\operatorname{sc}}(r) = \# \{ \zeta \in \mathcal{R}_g^{\operatorname{sc}} : |\zeta - n/2| \leq r \}.$$

It is the multiplicities of the scattering resonances that enter into the Poisson formula (1.5).

If (X, g) is strongly hyperbolic near infinity, it is shown in [2] that the following lower bounds, which take different forms depending on whether $\dim(X)$ is even or odd, hold. If $\dim(X)$ is even (i.e., n is odd), one has

$$(1.11) \quad N_g^{\operatorname{sc}}(r) \geq c |0\text{-Vol}(X, g)| r^{n+1}$$

for some $c > 0$ and r large (this result was already proved by Guillopé-Zworski in case $n = 1$, where $N_g(r) = N_g^{\operatorname{sc}}(r)$). On the other hand, if $\dim(X)$ is odd (i.e., n is even), the lower bound takes the form

$$(1.12) \quad N_g^{\operatorname{sc}}(r) \geq c |\chi(\bar{X})| r^{n+1}$$

where $c > 0$, r is sufficiently large. Although we consider the more general case of manifolds hyperbolic near infinity (i.e., dropping the “strongly”), this dichotomy will play an important role in our work.

The scattering resonances include both resolvent resonances and an additional set of singularities related to the conformal infinity. These singularities occur at $s = n/2 + k$ for $k = 1, 2, \dots$; at these points, the residue of the scattering operator $S_g(s)$ is an elliptic operator P_k on M with kernel having finite dimension d_k . The operators P_k are the GJMS operators [11] associated to the conformal infinity: their connection to scattering theory was elucidated by Graham and Zworski [12].

The precise relation between the respective multiplicities (1.3) and (1.10) for resolvent resonances and scattering resonances was partially established Guillopé-Zworski ($n = 1$) and Borthwick-Perry ($n \geq 1$) [3], and completed by Guillarmou [13]):

$$(1.13) \quad \nu_g(\zeta) = m_g(\zeta) - m_g(n - \zeta) + \sum_{k \in \mathbb{N}} (\mathbf{1}_{n/2-k}(\zeta) - \mathbf{1}_{n/2+k}(\zeta)) d_k.$$

Here $\mathbf{1}_t(s) = 1$ when $s = t$ and is zero elsewhere. This shows that the difference between the counting functions for resolvent resonances and the counting function

for scattering resonances comes from two sources: first, the finitely many ζ for which $n - \zeta$ corresponds to an eigenvalue of Δ_g and second, the numbers d_k . If we let $\mathcal{R}_g^{\text{GZ}}$ be the set $\{n/2 - k : k \in \mathbb{N}\}$ assigning multiplicity d_k to $\zeta = n/2 - k$, and

$$N_g^{\text{GZ}}(r) = \# \{ \zeta \in \mathcal{R}_g^{\text{GZ}} : |\zeta - n/2| \leq r \},$$

we have $N_g^{\text{sc}}(r) = N_g^{\text{GZ}}(r) + N_g(r)$ up to a finite error which does not affect upper and lower bounds for large r (this was first pointed out in the literature by Guillarmou and Naud [17]). Thus, in general, $N_g(r) \leq N_g^{\text{sc}}(r)$, so that lower bounds on $N_g^{\text{sc}}(r)$ do not imply lower bounds on $N_g(r)$.

On the one hand, it is reasonable to expect that the counting function $N_g(r)$, which is arguably a more natural counting function, obeys similar bounds. On the other hand, there are known examples where $N_g^{\text{GZ}}(r)$ saturates the lower bound (see also the remarks following Theorem 1.3 in [2]); indeed, if $X = \mathbb{H}^{n+1}$, real hyperbolic $(n + 1)$ -dimensional space, and n is even, then $N_g(r) = 0!$ (see Guillarmou-Naud [17] for further discussion). For this reason, one can only expect optimal lower bounds to hold in a “generic” sense.

We will say that the counting function $N_g(r)$ has *maximal order of growth* if $\rho = n + 1$, in correspondence to the known upper bounds. If $N_g(r)$ does not have maximal order of growth we will say that g is *resonance-deficient*. Our main result says that the counting function $N_g(r)$ has maximal order of growth for generic metrics in the following sense. Let us fix a manifold (X, g_0) , assumed hyperbolic near infinity, and a compact subset K of X . Let $\mathcal{G}(g_0, K)$ be the set of metrics g with $g = g_0$ outside K , and let $\mathcal{M}(g_0, K)$ be the subset of $\mathcal{G}(g_0, K)$ consisting of metrics for which $N_g(r)$ has maximal order of growth. Viewing metrics as sections of $\mathcal{C}^\infty(T^*X \otimes T^*X)$, we topologize these sets with the \mathcal{C}^∞ topology. This topology is compatible with norm resolvent convergence for the corresponding Laplacians.

Theorem 1.1. *Suppose that (X, g_0) is hyperbolic near infinity, and K is a compact subset of X . Then:*

- (i) *If n is odd, $\mathcal{M}(g_0, K)$ contains an open dense subset of $\mathcal{G}(g_0, K)$.*
- (ii) *If n is even, $\mathcal{M}(g_0, K)$ is a dense \mathcal{G}_δ set in $\mathcal{G}(g_0, K)$.*

Remark 1.2. If $n = 1$, it is known that $N_g^{\text{GZ}}(r) = 0$ so that $N_g^{\text{sc}}(r) = N_g(r)$ and $\mathcal{M}(g_0, K) = \mathcal{G}(g_0, K)$ for any $K \subset X$; see [19] and [1, section 8.5].

Remark 1.3. Theorem 1.1 gives a precise meaning to our assertion that optimal lower bounds hold for “generic” metrics.

Remark 1.4. For n odd, we actually prove a stronger statement, that resonance-deficient metrics can occur for at most one value of the zero-volume.

A key observation is that compact metric perturbations leave $N_g^{\text{GZ}}(r)$ unchanged since these resonances depend only on the conformal infinity of (X, g) ; thus it is natural to study the relative wave trace for the perturbed and unperturbed metrics.

The contents of this paper are as follows. In section 2, we consider a family of complexified metrics

$$g_z = (1 - z)g_0 + zg_1$$

for z in a small complex neighborhood of $[0, 1]$. Since this is not a family of Riemannian metrics, we study the analog of the Laplacian for g_z and its scattering operator. We then consider the relative wave trace between g_0 and a compactly supported perturbation g_1 in section 3, and prove the first part of Theorem 1.1.

Next, in section 4, we construct a compactly supported metric perturbation g_1 of g_0 obeying the optimal lower bound. Finally, in section 5, we extend the methods of [6] to prove the second part of Theorem 1.1.

Acknowledgment. The authors are grateful for support from the Mathematical Sciences Research Institute and the Banff International Research Station, where portions of the work were carried out.

2. INTERPOLATED LAPLACIAN AND RELATIVE SCATTERING MATRIX

Let (X, g_0) be conformally compact and hyperbolic near infinity, and g_1 another metric on X that agrees with g_0 outside some compact set $K \subset X$. For z in the rectangular region,

$$(2.1) \quad \Omega_\varepsilon := [-\varepsilon, 1 + \varepsilon] \times i[-\varepsilon, \varepsilon],$$

we define a bilinear form interpolating between the two metrics by

$$(2.2) \quad g_z = (1 - z)g_0 + zg_1.$$

Let P_{g_z} be the ‘‘Laplacian’’ associated to g_z in the formal sense,

$$P_{g_z} := -\frac{1}{\sqrt{\det g_z}} \partial_j [\sqrt{\det g_z} (g_z)^{jk}] \partial_k.$$

Assuming that ε is sufficiently small, $\det g_z$ will lie within the natural branch of the square root, and the coefficients of P_{g_z} will be analytic in z . With $z = a + ib$ for $a, b \in \mathbb{R}$, we regard P_{g_z} as an unbounded operator on $L^2(X, dg_a)$.

The goal of this section is to define an operator $S_{g_z}(s)$ as the scattering matrix associated to P_{g_z} . Since P_{g_z} is not self-adjoint, various facts need to be checked.

2.1. Analytic continuation of the resolvent of P_{g_z} . We first prove that the resolvent of P_{g_z} , written as $(P_{g_z} - s(n - s))^{-1}$, admits an analytic continuation in s .

Lemma 2.1. *Assuming ε is sufficiently small, there exist $a_\varepsilon, C_\varepsilon$ independent of z , such that for $\Re s > a_\varepsilon \geq n$, the operator $P_{g_z} - s(n - s)$ is invertible and the inverse satisfies*

$$\|(P_{g_z} - s(n - s))^{-1}\|_{L^2(X, dg_a)} \leq \frac{C_\varepsilon}{\Re(s)}.$$

Proof. Since $P_{g_a} = \Delta_{g_a}$, the Laplacian of an actual metric g_a , $R_{g_a}(s)$ is analytic for $\Re(s) > n$. Consider the simple identity

$$(2.3) \quad (P_{g_z} - s(n - s))R_{g_a}(s) = I + (P_{g_z} - P_{g_a})R_{g_a}(s).$$

Since $P_{g_z} - P_{g_a}$ is a compactly supported second order differential operator and $R_{g_a}(s)$ has order -2 , the operator norm of $(P_{g_z} - P_{g_a})R_{g_a}(s)$ may be estimated for all $\Re(s)$ sufficiently large by the supremum of the coefficients of $P_{g_z} - P_{g_a}$. These coefficients are clearly $O(\varepsilon)$, so by choosing ε small we may assume

$$\|(P_{g_z} - P_{g_a})R_{g_a}(s)\| \leq \frac{1}{2} \text{ for all } \Re(s) > a_\varepsilon.$$

This shows that the right side of (2.3) is invertible, and hence that $P_{g_z} - s(n - s)$ is invertible. The norm estimate on the inverse then follows immediately from the

Neumann series estimate,

$$\begin{aligned} \|[I + (P_{g_z} - P_{g_a})R_{g_a}(s)]^{-1}\| &\leq \sum_{l=0}^{\infty} \|(P_{g_z} - P_{g_a})R_{g_a}(s)\|^l \\ &\leq 2 \quad \text{for } \Re(s) > a_\varepsilon, \end{aligned}$$

and the standard resolvent estimate on $R_{g_a}(s)$, which for $\Re(s) \geq n$ gives

$$\|R_{g_a}(s)\| \leq \frac{1}{|s(n-s)|}.$$

□

Since P_{g_z} agrees with Δ_{g_0} outside K , Lemma 2.1 leads almost immediately to a proof of analytic continuation of the resolvent of P_{g_z} . Recall that x is a boundary defining function for the boundary $\partial\bar{X}$, and let \mathcal{B}_N denote the bounded operators from $x^N L^2(X, dg_a) \rightarrow x^{-N} L^2(X, dg_a)$.

Proposition 2.2. *The resolvent $R_{g_z}(s) := (P_{g_z} - s(n-s))^{-1}$, which by Lemma 2.1 is defined for $z \in \Omega_\varepsilon$ and $\Re(s) > a_\varepsilon$, admits for any $N > 0$ a finitely meromorphic continuation as a \mathcal{B}_N -valued function of s to the region $\Re(s) > -N + \frac{n}{2}$. For $(z, s) \in \Omega_\varepsilon \times (\Re(s) > n/2)$, $R_z(s)$ is meromorphic in two variables as a \mathcal{B}_N operator-valued function.*

Proof. The resolvent $R_{g_a}(s)$ serves as a suitable parametrix for $R_{g_z}(s)$ near the boundary. Let $\chi, \chi_0, \chi_1 \in C^\infty(X)$ be cutoff functions vanishing in some neighborhood of K and equal to 1 in some neighborhood of $\partial\bar{X}$, such that $\chi = 1$ on the support of χ_0 and $\chi_1 = 1$ on the support of χ . Then for large $s_0 > 0$ we set

$$(2.4) \quad M(s) = (1 - \chi_0)R_{g_z}(s_0)(1 - \chi) + \chi_1 R_{g_a}(s)\chi.$$

Then, using the facts that $\chi_1\chi = \chi$ and $(1 - \chi)(1 - \chi_0) = (1 - \chi)$, we obtain

$$(2.5) \quad (P_{g_z} - s(n-s))M(s) = I - K_1(s) - K_2(s),$$

where

$$K_1(s) := [\Delta_{g_0}, \chi_0]R_{g_z}(s_0)(1 - \chi) + (s_0(n - s_0) - s(n - s))(1 - \chi_0)R_{g_z}(s_0)(1 - \chi)$$

and

$$K_2(s) := [\Delta_{g_0}, \chi_1]R_{g_a}(s)\chi.$$

The error term $K_1(s)$ is a compactly supported pseudodifferential operator of order -2 , whose operator norm may be made arbitrarily small by choosing s_0 large, according to Lemma 2.1. The error term $K_2(s)$ has a smooth kernel contained in $x^\infty x'^s C^\infty(X \times X)$. For $N > 0$, $K_2(s)$ is a compact operator on $x^N L^2(X, dg_a)$ for $\Re s > -N + \frac{n}{2}$. Its norm may be made arbitrarily small by choosing $\Re(s)$ large using to the standard resolvent estimate on $R_{g_a}(s)$.

Since $K_1(s)$ and $K_2(s)$ are meromorphic both in z and in s , the analytic Fredholm theorem thus applies to show that $I - K_1(s) - K_2(s)$ is invertible meromorphically on $\rho^N L^2(X, dg_a)$ for $z \in \Omega_\varepsilon$ and $\Re(s) > -N + \frac{n}{2}$. □

2.2. Upper bounds on the resonance counting function for P_{g_z} . Proposition 2.2 allows us to define \mathcal{R}_{g_z} as the set of resonances ζ of $R_{g_z}(s)$, with multiplicities counted by

$$m_z(\zeta) := \text{rank Res}_\zeta R_{g_z}(s).$$

The associated resonance counting function is

$$N_{g_z}(r) := \#\{\zeta \in \mathcal{R}_{g_z} : |\zeta| \leq r\}.$$

For real z , polynomial bounds on the growth of $N_{g_z}(r)$ were proven in [18], and an optimal upper bound on the growth of $N_{g_z}(r)$ was proven by Cuevas-Vodev [8] and Borthwick [2]. We need to extend this bound to $z \in \Omega_\varepsilon$.

Proposition 2.3. *For $\varepsilon > 0$ sufficiently small, there exists C_ε independent of $z \in \Omega_\varepsilon$ such that*

$$N_{g_z}(r) \leq C_\varepsilon r^{n+1}.$$

Proof. In the proofs cited above, the interior metric enters only in the interior parametrix term, i.e., the first term on the right in (2.4). Most of the work goes into estimation of the boundary terms, and these results apply immediately to P_{g_z} because $P_{g_z} = \Delta_{g_0}$ on $X - K$.

In the argument from Cuevas-Vodev, the only estimate required of the interior term is [8, eq. (2.24)], an estimate on the singular values the operator $K_1(s)$ defined above. These estimates depend only on the fact that $K_1(s)$ is compactly supported and of order -2 . For ε sufficiently small, P_{g_z} will be uniformly elliptic for $z \in \Omega_\varepsilon$, and so $R_{g_z}(s)$ will have order -2 and the required estimates on $K_1(s)$ can be done uniformly in z . The proof of [8, Prop. 1.2] then gives a bound

$$\#\{\zeta \in \mathcal{R}_{g_z} : |\zeta| \leq r, \arg(\zeta - \frac{n}{2}) \in [-\pi + \varepsilon, \pi - \varepsilon]\} \leq C_\varepsilon r^{n+1}.$$

To fill in the missing sector containing the negative real axis, we apply the argument from Borthwick [2]. Here the interior parametrix enters only in the proof of [2, Lemma 5.2]. In the original version, the standard resolvent estimate was used in the form $\|R_{g_z}(n-s)\| = O(1)$ for $\Re s \leq 0$. For $R_{g_z}(n-s)$ this must be replaced by the estimate from Lemma 2.1, which gives $\|R_{g_z}(n-s)\| = O(1)$ for $\Re s < n - a_\varepsilon$. The result is that we have

$$\#\{\zeta \in \mathcal{R}_{g_z} : |\zeta| \leq r, \arg(\zeta - n + a_\varepsilon) \in [\frac{\pi}{2} + \varepsilon, \frac{3\pi}{2} - \varepsilon]\} \leq C_\varepsilon r^{n+1}.$$

Since the two estimates obtained cover all but a compact region, the result follows. \square

2.3. The scattering matrix associated with P_{g_z} . The meromorphic continuation of $R_{g_z}(s)$ allows us to define the associated scattering matrix $S_{g_z}(s)$ exactly as in (1.7)-(1.8). Scattering multiplicities are defined by

$$\nu_{g_z}(\zeta) := -\text{tr}[\text{Res}_\zeta \tilde{S}'_{g_z}(s) \tilde{S}_{g_z}(s)^{-1}],$$

where

$$\tilde{S}_{g_z}(s) := \frac{\Gamma(s - \frac{n}{2})}{\Gamma(\frac{n}{2} - s)} S_{g_z}(s).$$

Since the relation between scattering poles and resonances depends only on the boundary structure of the resolvent, it carries over immediately to $S_{g_z}(s)$,

$$(2.6) \quad \nu_{g_z}(\zeta) = m_z(\zeta) - m_z(n - \zeta) + \sum_{k \in \mathbb{N}} \left(\mathbf{1}_{n/2-k}(\zeta) - \mathbf{1}_{n/2+k}(\zeta) \right) d_k,$$

where

$$d_k = \dim \ker P_k$$

with

$$P_k = \tilde{S}_{g_0}(\frac{n}{2} + k).$$

Applying $R_{g_z}(s)$ to (2.5) from the left, we obtain the identity

$$R_{g_z}(s) = M(s) + R_{g_z}(s)(K_1(s) + K_2(s))$$

By taking the boundary limits of this formula as the boundary defining functions $x, x' \rightarrow 0$, we obtain some useful relations. The Poisson operators associated to P_{g_z} and Δ_{g_0} are related by

$$(2.7) \quad E_{g_z}(s) = E_{g_0}(s) + R_{g_z}(s)[\Delta_{g_0}, \chi_1]E_{g_0}(s),$$

and for the scattering matrices we have

$$(2.8) \quad S_{g_z}(s) = S_{g_0}(s) + E_{g_z}(s)^t[\Delta_{g_0}, \chi_1]E_{g_0}(s).$$

The latter equation shows that $S_{g_z}(s)$ and $S_{g_0}(s)$ differ by a smoothing operator on $\partial\bar{X}$. This shows in particular that the relative scattering matrix $S_{g_z}(s)S_{g_0}(s)^{-1}$ is determinant class. In fact, by the identity $E(s)S(s)^{-1} = -E(n-s)$, the relative scattering matrix is given explicitly by

$$(2.9) \quad S_{g_z}(s)S_{g_0}(s)^{-1} = I - E_{g_z}(s)^t[\Delta_{g_0}, \chi_1]E_{g_0}(n-s)$$

We can exploit these relationships further by substituting the transpose of (2.7) into (2.9). This yields

$$(2.10) \quad \begin{aligned} S_{g_z}(s)S_{g_0}(s)^{-1} &= I - E_{g_0}(s)^t[\Delta_{g_0}, \chi_1]E_{g_0}(n-s) \\ &\quad - ([\Delta_{g_0}, \chi_1]E_{g_0}(s))^t R_{g_z}(s)[\Delta_{g_0}, \chi_1]E_{g_0}(n-s). \end{aligned}$$

The point of this formula is that the dependence on g_z is isolated in the $R_{g_z}(s)$ term. It also shows that $S_{g_z}(s)S_{g_0}(s)^{-1}$ is a meromorphic function of z and s since the same is true of $R_{g_z}(s)$. We will use it later to estimate $S_{g_z}(s)S_{g_0}(s)^{-1}$ in terms of the difference in the metrics. Note that, since $R_{g_z}(s)$ is meromorphic in $\Omega_\varepsilon \times \mathbb{C}$, so is $S_{g_z}(s)$.

Let $H_z(s)$ denote the Hadamard product over the resonance set \mathcal{R}_{g_z} :

$$(2.11) \quad H_z(s) := \prod_{\zeta \in \mathcal{R}_{g_z}} E\left(\frac{s}{\zeta}, n+1\right),$$

where

$$E(u, p) := (1-u) \exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}\right).$$

The relative scattering determinant may be defined as

$$\sigma_{g_z, g_0}(s) := \det[S_{g_z}(s)S_{g_0}^{-1}(s)].$$

Proposition 2.4. *The relative scattering determinant admits a factorization*

$$(2.12) \quad \sigma_{g_z, g_0}(s) = e^{q(s)} \frac{H_z(n-s)}{H_z(s)} \frac{H_0(s)}{H_0(n-s)},$$

where $q(s)$ is a polynomial of degree at most $n+1$.

Proof. Let $A(s)$ be the auxiliary operator introduced in [2, §3], defined so that $S_{g_z}(s) - A(s)$ is smoothing. Note that the construction of $A(s)$ depends only on the metric in a neighborhood of $\partial\bar{X}$ and so the same $A(s)$ works for any of the “metrics” g_z . We set

$$\vartheta_z(s) := \det S_{g_z}(n-s)A(s).$$

The arguments in [2, §6] apply immediately to show that $\vartheta_z(s)$ is a ratio of entire functions of bounded order. Furthermore

$$\det S_{g_z}(s)S_{g_0}(s)^{-1} = \frac{\vartheta_0(s)}{\vartheta_z(s)}.$$

In computing the divisor of $\vartheta_0(s)/\vartheta_z(s)$, the terms coming from $A(s)$ cancel, and we find, by the definition of $\nu_{g_z}(\zeta)$,

$$\operatorname{Res}_\zeta \frac{\vartheta'_z}{\vartheta_z}(s) - \operatorname{Res}_\zeta \frac{\vartheta'_0}{\vartheta_0}(s) = -\nu_{g_z}(\zeta) + \nu_{g_0}(\zeta).$$

Hence the relation (2.6) shows that both sides of (2.12) have the same divisor. We have thus proven (2.12) with $q(s)$ some polynomial of unknown degree.

To control the degree, we use Lemma 2.1 to adapt the proof of [2, Lemma 5.2], just as we did above, to prove for $\Re(s) < n - a_\varepsilon$ that

$$|\vartheta_z(s)| < e^{C_{\eta,\varepsilon}(s)^{n+1}},$$

provided $d(s, -\mathbb{N}_0) > \eta$. Since we can write

$$\vartheta_z(s) = e^{-q(s)} \frac{H_0(n-s)}{H_0(s)} \frac{H_z(s)}{H_z(n-s)} \vartheta_0(s),$$

and the Hadamard products have order $n+1$, this shows that $|q(s)| \leq C|s|^{n+1+\delta}$ in the half-plane $\Re(s) < n - a_\varepsilon$ for any $\delta > 0$. Hence the degree of $q(s)$ is at most $n+1$. \square

Define the meromorphic function $\Upsilon_z(s)$ by

$$\Upsilon_z(s) = (2s-n)0\text{-Tr}[R_{g_z}(s) - R_{g_z}(1-s)],$$

for $s \notin \mathbb{Z}/2$. The connection between $\Upsilon_z(s)$ and the logarithmic derivative of the scattering determinant established by Patterson-Perry [29, Prop. 5.3 and Lemma 6.7] depends only on the structure of model neighborhoods near infinity, and so carries over to our case without alteration. This yields the following Birman-Krein type formula:

Proposition 2.5. *For $s \notin \mathbb{Z}/2$ we have the meromorphic identity,*

$$-\frac{d}{ds} \log \sigma_{g_z, g_0}(s) = \Upsilon_z(s) - \Upsilon_0(s).$$

For a real (so that g_a is an actual metric), we define the relative volume

$$V_{\text{rel}}(a) = \operatorname{Vol}(K, g_a) - \operatorname{Vol}(K, g_0).$$

We can derive asymptotics from Proposition 2.5 as in Borthwick [2, Thm. 10.1]. Furthermore, the restriction to metrics strongly hyperbolic near infinity in [2] can be relaxed here because we are only interested in the *relative* scattering determinant.

Corollary 2.6. For $a \in [-\varepsilon, 1 + \varepsilon]$, as $\xi \rightarrow +\infty$,

$$\log \sigma_{g_a, g_0}(\frac{n}{2} + i\xi) = c_n V_{\text{rel}}(a) \xi^{n+1} + \mathcal{O}(\xi^n).$$

where

$$c_n = -2\pi i \frac{(4\pi)^{-(n+1)/2}}{\Gamma(\frac{n+3}{2})}.$$

3. LOWER BOUNDS FROM THE RELATIVE WAVE TRACE

If the dimension $n + 1$ is even (n odd), then we can deduce a lower bound on the resolvent resonances by using a relative wave trace to cancel the conformal Graham-Zworski scattering poles (the d_k terms in Poisson formula [2, Thm. 1.2]).

Let (X, g_0) be conformally compact and hyperbolic near infinity, and g_1 another metric that agrees with g_0 outside some compact set $K \subset X$. By the functional calculus, $\Upsilon_a(\frac{n}{2} + i\xi)$ is essentially the Fourier transform of the continuous part of the wave 0-trace (see [2, Lemma 8.1]). By Propositions 2.4 and 2.5 we can write

$$\Upsilon_1(s) - \Upsilon_0(s) = \partial_s \log \left[e^{q(s)} \frac{H_1(s)}{H_1(n-s)} \frac{H_0(n-s)}{H_0(s)} \right]$$

Taking the Fourier transform just as in the proof of [2, Thm. 1.2] then gives

Theorem 3.1. For (X, g_0) conformally compact and hyperbolic near infinity, and g_1 a compactly supported perturbation, we have

$$\begin{aligned} & 0\text{-Tr} \left[\cos \left(t\sqrt{\Delta_{g_1} - n^2/4} \right) \right] - 0\text{-Tr} \left[\cos \left(t\sqrt{\Delta_{g_0} - n^2/4} \right) \right] \\ &= \frac{1}{2} \sum_{\zeta \in \mathcal{R}_{g_1}} e^{(\zeta - n/2)|t|} - \frac{1}{2} \sum_{\zeta \in \mathcal{R}_{g_0}} e^{(\zeta - n/2)|t|}, \end{aligned}$$

in the sense of distributions on $\mathbb{R} - \{0\}$.

(Note that [2, Thm. 1.2] required a metric strongly hyperbolic near infinity; we may drop that restriction here because we are dealing with the difference of two wave traces.)

Theorem 3.1 applies in any dimension, but it only gives a lower bound on resonances when the singularity on the wave trace side spreads out beyond $t = 0$. The following Corollary requires $n + 1$ even and a nonzero relative volume between the two metrics.

Corollary 3.2. Assume that $n + 1$ is even and g_0, g_1 are metrics as above. There is a constant $c > 0$ such that

$$N_{g_0}(r) + N_{g_1}(r) \geq c |\text{Vol}(K, g_1) - \text{Vol}(K, g_0)| r^{n+1}.$$

Proof. For $\phi \in C_0^\infty(\mathbb{R}_+)$ and $\lambda > 0$ we can apply [2, Lemma 9.2] to obtain from Theorem 3.1 the asymptotic

$$\left| \sum_{\zeta \in \mathcal{R}_{g_1}} \widehat{\phi}(i(\zeta - \frac{n}{2})/\lambda) - \sum_{\zeta \in \mathcal{R}_{g_0}} \widehat{\phi}(i(\zeta - \frac{n}{2})/\lambda) \right| = c_n |\text{Vol}(K, g_1) - \text{Vol}(K, g_0)| \lambda^{n+1} + \mathcal{O}(\lambda^{n-1}),$$

as $\lambda \rightarrow \infty$. Since ϕ is compactly supported, its Fourier transform satisfies analytic estimates,

$$|\widehat{\phi}(\xi)| \leq C_m (1 + |\xi|)^{-m},$$

for $m \in \mathbb{N}$. Thus for λ sufficiently large and setting $m = n + 2$,

$$\begin{aligned} c_n |\text{Vol}(K, g_1) - \text{Vol}(K, g_0)| \lambda^{n+1} &\leq \sum_{\zeta \in \mathcal{R}_{g_0} \cup \mathcal{R}_{g_1}} |\widehat{\phi}(i(\zeta - \frac{n}{2})/\lambda)| \\ &\leq C \sum_{\zeta \in \mathcal{R}_{g_0} \cup \mathcal{R}_{g_1}} (1 + |\zeta|/\lambda)^{-n-2}, \end{aligned}$$

Then, if we let $M(r) = N_{g_0}(r) + N_{g_1}(r)$, we have

$$\begin{aligned} c_n |\text{Vol}(K, g_1) - \text{Vol}(K, g_0)| \lambda^{n+1} &\leq C \int_0^\infty (1 + r/\lambda)^{-n-2} dM(r) \\ &\leq C \int_0^\infty (1 + r)^{-n-3} M(\lambda r) dr. \end{aligned}$$

Splitting the integral at b and using the upper bound from Proposition 2.3 to control the $[b, \infty)$ piece then yields

$$c_n |\text{Vol}(K, g_1) - \text{Vol}(K, g_0)| \lambda^{n+1} \leq CM(\lambda b) + C\lambda^{n+1}b^{-1}.$$

Taking b sufficiently large completes the proof. \square

We conclude this section with:

Proof of part (i) of Theorem 1.1: Suppose that $\dim(X)$ is even. If $\mathcal{G}(g_0, K)$ contains resonance-deficient metrics, then we may redefine g_0 to assume that this background metric is resonance-deficient. Observe that for a fixed compact subset K of X , the function

$$\begin{aligned} \mathcal{G}(g_0, K) &\mapsto \mathbb{R} \\ g &\rightarrow 0\text{-Vol}(X, g) \end{aligned}$$

is continuous. Moreover, if we fix $g \in \mathcal{G}(g_0, K)$ and $\varphi \in \mathcal{C}_0^\infty(K)$, and consider the family

$$g_t = e^{t\varphi} g,$$

we have

$$\left. \frac{d}{dt} \right|_{t=0} (0\text{-Vol}(X, g_t)) = \int \varphi dg$$

which is nonzero for any nonzero, nonnegative $\varphi \in \mathcal{C}_0^\infty(K)$.

By continuity,

$$\mathcal{S} = \{g \in \mathcal{G}(g_0, K) : 0\text{-Vol}(X, g) \neq 0\text{-Vol}(X, g_0)\}$$

is open in $\mathcal{G}(g_0, K)$. By the conformal perturbation argument above, \mathcal{S} is also dense in $\mathcal{G}(g_0, K)$. It follows from Corollary 3.2 that $\mathcal{S} \subset \mathcal{M}(g_0, K)$, proving Theorem 1.1(i). \square

4. A METRIC PERTURBATION WITH OPTIMAL ORDER OF GROWTH

In this section, we prove:

Theorem 4.1. *Suppose that (X, g_0) is hyperbolic near infinity and $\dim(X) = n + 1$. Suppose that $N_{g_0}(r) = o(r^{n+1})$ as $r \rightarrow \infty$, let $x_0 \in X$. There is a Riemannian metric g_1 on X with the following properties: $g_1 = g_0$ outside $B(x_0, 3)$, and $N_{g_1}(r) \geq Cr^{n+1}$ for a strictly positive constant C and sufficiently large r .*

The hypothesis of Theorem 4.1 implies that the distribution $u_0(t)$ on $\mathbb{R} \setminus \{0\}$ defined by

$$(4.1) \quad u_0(t) = \frac{1}{2} \sum_{\xi \in \mathcal{R}_{g_0}} e^{(\zeta - n/2)|t|},$$

where \mathcal{R}_0 is the set of resolvent resonances for g_0 , satisfies

$$(4.2) \quad |\widehat{\varphi}u_0(\lambda)| = o(\lambda^n)$$

for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^+)$. Let

$$(4.3) \quad u_1(t) = \sum_{\xi \in \mathcal{R}_{g_1}} e^{(\zeta - n/2)|t|}.$$

where \mathcal{R}_1 is the set of resolvent resonances for g_1 . Following ideas of Sjöstrand-Zworski [32], we will construct a perturbed metric which, geometrically, attaches a large sphere to X at x_0 , and use wave trace estimates on $u_1 - u_0$ and the following Tauberian theorem [32, p. 848] to prove a lower bound on the counting function for the resonances of the perturbed metric.

Theorem 4.2. [32] *Let $u_1 \in \mathcal{D}'(\mathbb{R})$ be the distribution associated with the resolvent resonance set \mathcal{R}_{g_1} as in (4.3). Suppose that for some constants $b, d > 0$ and every $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^+)$ supported in a sufficiently neighborhood of d with $\varphi(d) = 1$ and $\widehat{\varphi}(\tau) \geq 0$, we have*

$$|\widehat{\varphi}u_1(\lambda)| \geq (b - o(1))\lambda^n$$

as $\lambda \rightarrow +\infty$. Then, the resonance counting function satisfies

$$N_{g_1}(r) \geq (B - o(1))r^{n+1}, \quad B = b/(\pi(n+1)).$$

Thus, we need to choose g_1 so that $|\widehat{\varphi}u_1(\lambda)| \geq C\lambda^n$ as $\lambda \rightarrow +\infty$. By (4.2) it suffices to prove the same estimate for $u_1 - u_0$. It follows from the relative Poisson formula, Theorem 3.1, that $u_1(t) - u_0(t)$ is a difference of wave traces.

Sjöstrand and Zworski used this idea in the Euclidean setting to construct scattering metrics which are Euclidean near infinity and whose resonance counting function has optimal order of growth. In our setting, the background metric is more complicated, so we begin with some perturbative estimates on the wave trace.

Let $x_0 \in X$ and denote by $B(x_0, 3)$ the ball of radius 3 in the *unperturbed* metric. We consider metrics g_0 and g_1 on a manifold X so that $g_1 = g_0$ on $X \setminus B(x_0, 3)$ and both metrics are hyperbolic near infinity. We will make a specific choice of g_1 later. We denote by Δ_0 and Δ_1 the respective positive Laplace-Beltrami operators and set

$$Q_0 = \begin{pmatrix} 0 & I \\ -(\Delta_0 - n^2/4) & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & I \\ -(\Delta_1 - n^2/4) & 0 \end{pmatrix},$$

where $n^2/4$ is the bottom of the continuous spectrum. These operators are the infinitesimal generators of wave groups $U_0(t) = \exp(tQ_0)$ and $U_1(t) = \exp(tQ_1)$ acting on the Hilbert spaces of initial data (v_0, v_1) of finite energy, defined as follows. Let (Y, g) denote either (X, g_0) or (X, g_1) . Let \mathcal{H} denote the completion of $\mathcal{C}_0^\infty(Y) \oplus \mathcal{C}_0^\infty(Y)$ in the norm

$$\|(v_0, v_1)\|_Y = \|\nabla v_0\| + \|v_1\|$$

where $\|\cdot\|$ denotes the $L^2(Y, g)$ norm. Letting $\dot{H}^1(Y, g)$ denote the completion of $\mathcal{C}_0^\infty(Y)$ in the norm $\|\nabla(\cdot)\|$ modulo constants, we have $\mathcal{H} = \dot{H}^1(Y, g) \oplus L^2(Y, g)$.

An important remark (see, for example, [25, Chapter IV, Lemma 1.1]) is that $\dot{H}^1(Y, g) \subset L^2_{\text{loc}}(Y, g)$ and that the Sobolev bound

$$\left(\int |v|^{2(n+1)/(n-1)} dg \right)^{(n-1)/2(n+1)} \leq c \left(\int |\nabla v|^2 \right)^{1/2}$$

holds (recall $\dim Y = n + 1$). The wave groups $U_0(t)$ and $U_1(t)$ act as unitary groups on their respective Hilbert spaces.

To make perturbative estimates, it is convenient to use the natural unitary map $J : L^2(X, dg_0) \rightarrow L^2(X, dg_1)$ and define $U(t) = J^* U_1(t) J$. The operators $U(t)$ are a unitary group on \mathcal{H}_0 with infinitesimal generator

$$Q = \begin{pmatrix} 0 & I \\ -(\Delta - n^2/4) & 0 \end{pmatrix}$$

where Δ is a second-order elliptic differential operator with $\Delta = \Delta_0$ on functions with support contained in $X \setminus B(x_0, 3)$.

We will be interested in Fourier transforms of the wave trace of the form (4.2) where φ is localized near the period T of a closed geodesic. Let $\varphi \in \mathcal{C}_0^\infty([-1, 1])$ with $\varphi(0) = 1/(2\pi)$ and $\widehat{\varphi}(\tau) \geq 0$, and define

$$(4.4) \quad \varphi_{\varepsilon, T}(t) = \varphi\left(\frac{t-T}{\varepsilon}\right).$$

Let $D(t)$ be the distribution

$$D(t) = 0 - \text{Tr}(U(t) - U_0(t))$$

and consider the Fourier transform

$$(4.5) \quad \Phi(\lambda) = \int e^{-i\lambda t} \varphi_{\varepsilon, T}(t) D(t) dt.$$

which is the difference of $\widehat{\varphi_{\varepsilon, T} u_1}$ and $\widehat{\varphi_{\varepsilon, T} u_0}$. We will first isolate the dominant term in $\Phi(\lambda)$ for a arbitrary compactly supported perturbation, and then make a specific choice of g_1 that produces the desired $\mathcal{O}(\lambda^n)$ growth.

In what follows, it will be important to microlocalize in the unit cosphere bundle S^*X . We denote by $\Pi : S^*X \rightarrow X$ the canonical projection. For $(x, \xi) \in S^*X$, we denote by $\gamma_t(x, \xi)$ the unit speed geodesic passing through (x, ξ) at time zero. Unless otherwise stated, the geodesics will be defined with respect to the perturbed metric on X . Note that, on $X \setminus B(x_0, 3)$, these geodesics coincide with those of g_0 .

The first lemma allows us to localize the wave trace near the perturbation up to controlled errors. Let $\psi \in \mathcal{C}_0^\infty(X)$ with

$$(4.6) \quad \psi(x) = \begin{cases} 1 & d(x_0, x) < 4 \\ 0 & d(x_0, x) > 6 \end{cases}$$

where $d(\cdot, \cdot)$ is the distance in the unperturbed metric g_0 .

Lemma 4.3. *The asymptotic formula*

$$\Phi(\lambda) = \int e^{-i\lambda t} \varphi_{\varepsilon, T}(t) \text{Tr}[(U(t) - U_0(t))\psi] dt + \mathcal{O}(T\lambda^n)$$

holds as $\lambda \rightarrow \infty$.

Proof. First, by finite propagation speed, it follows that $U(t)f = U_0(t)f$ for any $t \in \text{supp } \varphi_{\varepsilon, T}$ and f with support a distance at least $2T$ from $B(x_0, 3)$. Hence, if

$$\chi_T(x) = \begin{cases} 1 & d(x_0, x) < 2T \\ 0 & d(x_0, x) > 3T \end{cases}$$

(where $d(\cdot, \cdot)$ is the distance in the unperturbed metric, and $T > 2$ say), we have

$$\Phi(\lambda) = \int e^{-i\lambda t} \varphi_{\varepsilon, T}(t) \text{Tr}[(U(t) - U_0(t))\chi_T] dt.$$

It suffices to show that

$$(4.7) \quad \int e^{-i\lambda t} \varphi_{\varepsilon, T}(t) \text{Tr}[(U(t) - U_0(t))(1 - \psi)\chi_T] dt = \mathcal{O}(T\lambda^n)$$

since $\psi\chi_T = \psi$. Let $C \in \Psi_{\text{phg}}^0(X)$ be a pseudodifferential operator with the following properties:¹

$$(4.8) \quad \text{S-ES}(C) \subset \{(x, \xi) \in S^*X : \Pi\gamma_t(x, \xi) \in B(x_0, 5), \exists t \in [-1, 4T]\},$$

$$(4.9) \quad \text{S-ES}(I - C) \subset \{(x, \xi) \in S^*X : \Pi\gamma_t(x, \xi) \notin B(x_0, 4), \forall t \in [-1, 4T]\}$$

where I denotes the identity operator, and the geodesics and balls are understood to be defined with respect to g_0 . We split

$$(U(t) - U_0(t))(1 - \psi)\chi_T = G_1(t) + G_2(t)$$

where

$$\begin{aligned} G_1(t) &= (U(t) - U_0(t))(I - C)(1 - \psi)\chi_T, \\ G_2(t) &= (U(t) - U_0(t))C(1 - \psi)\chi_T. \end{aligned}$$

First, we claim that $G_1(t)$ is a smoothing operator for $t \in \text{supp } (\varphi_{\varepsilon, T})$. To see this, note that $G_1(0) = 0$ so by the Fundamental Theorem of Calculus

$$G_1(t) = \int_0^t U(t-s)(Q - Q_0)U_0(s)(I - C)\chi_T(1 - \psi) ds.$$

Note that $Q - Q_0 = 0$ outside $B(x_0, 3)$, and let $\theta \in C^\infty(X)$ with

$$\theta(x) = \begin{cases} 1 & x \in B(x_0, 7/2), \\ 0 & x \notin B(x_0, 15/4). \end{cases}$$

(where again the balls are defined with respect to g_0). By the propagation of singularities and (4.9), the operator $\theta U_0(s)(I - C)$ has a smooth kernel for all $t \in [0, 2T]$. Combining these observations we see that

$$G_1(t) = \int_0^t U(t-s)(Q - Q_0)\theta U_0(s)(I - C)\chi_T(1 - \psi) ds$$

is a smoothing operator for $t \in [0, 2T]$. It follows that

$$(4.10) \quad \int e^{-i\lambda t} \varphi_{\varepsilon, T}(t) \text{Tr } G_1(t) dt = \mathcal{O}(\lambda^{-\infty}).$$

¹See Appendix A for the definition of the essential support S-ES of a pseudodifferential operator.

FIGURE 1. X_R is constructed by gluing a sphere of radius R to $X \setminus B_X(x_0, 1)$.

Next, we consider $G_2(t)$. The operator $C_T = C(1 - \psi)\chi_T$ has $\text{S-ES}(C_T)$ contained in a subset of S^*X having volume $\mathcal{O}(T)$ (compare Lemma 4.6 below; here volume is unambiguously given by g_0 since $\pi(\text{S-ES}(C_T))$ lies away from the metric perturbation). We can then deduce that

$$(4.11) \quad \int e^{-i\lambda t} \varphi_{\varepsilon, T}(t) \text{Tr} G_2(t) dt = \mathcal{O}(T\lambda^n)$$

by applying Lemma A.1 to the two respective terms involving $U(t)$ and $U_0(t)$. The estimate (4.7) follows from (4.10) and (4.11). \square

Next, we note:

Lemma 4.4. *The estimate*

$$\int e^{-i\lambda t} \varphi_{\varepsilon, T}(t) \text{Tr} [U_0(t)\psi] dt = \mathcal{O}_{\varepsilon, \psi}(\lambda^n)$$

holds.

Proof. An immediate consequence of Lemma A.1 with $B = \psi$. \square

Combining Lemmas 4.3 and 4.4, we have shown that

$$(4.12) \quad \Phi(\lambda) = \Phi_1(\lambda) + \mathcal{O}_{\varepsilon, \psi}(T\lambda^n).$$

where

$$\Phi_1(\lambda) = \int e^{-i\lambda t} \varphi_{\varepsilon, T}(t) \text{Tr} [U(t)\psi] dt.$$

We now make a choice of g_1 so that (X, g_1) is isometric to a manifold (X_R, g_R) defined as follows. Roughly, X_R is X with a ball excised, and a large Euclidean sphere glued in analogy to the construction in [32]. More precisely, denote by $\mathbb{S}^m(R)$ the Euclidean sphere of radius R and dimension m with the usual metric. Pick a point $x_0 \in X$ and $x_1 \in \mathbb{S}^{n+1}(R)$. The manifold X_R consists of $X \setminus B_X(x_0, 1)$ together with a cylindrical neck $N = \mathbb{S}^n(1) \times [0, 1]$ that connects $X \setminus B(x_0, 1)$ to $\mathbb{S}^{n+1}(R) \setminus B_{\mathbb{S}^{n+1}(R)}(x_1, 1)$ (we make the natural identification between $\mathbb{S}^n(1)$ and $\partial B(x_0, 1) \subset X$ on the one hand, and $\mathbb{S}^n(1)$ and $\partial B(x_1, 1) \subset \mathbb{S}^{n+1}(R)$ on the other). Thus

$$X_R = (X \setminus B_X(x_0, 1)) \sqcup N \sqcup (\mathbb{S}^{n+1}(R) \setminus B_{\mathbb{S}^{n+1}(R)}(x_1, 1)).$$

We put a smooth metric g_R on X_R which coincides with the standard metric on the sphere on $\mathbb{S}^{n+1}(R) \setminus B_{\mathbb{S}^{n+1}(R)}(x_1, 2)$, and the original metric g_0 on $X \setminus B(x_0, 3)$. There is a natural diffeomorphism $f : X \rightarrow X_R$ and we take $g_1 = f^*g_R$.

With this choice of perturbation, we wish to show that $\Phi(\lambda)$ has essentially the same behavior as the wave trace on the sphere. We now make the choice $T = 2\pi R$ to localize near the periods of geodesics on the sphere. Let $U_S(t)$ denote the wave group on $\mathbb{S}^{n+1}(R)$, and define

$$(4.13) \quad \Phi_0(\lambda) = \int \varphi_{\varepsilon, 2\pi R}(t) \text{Tr} [U_S(t)] dt$$

Recall (see for example [9], section 3):

Lemma 4.5. *There is a strictly positive constant c_n depending only on n so that*

$$\Phi_0(\lambda) = c_n R^n \lambda^n + \mathcal{O}(\lambda^{n-1}).$$

Proof. This follows from the fact that the leading singularity of $U_S(t)$ at $t = 2\pi R$ is $c_n R^n \delta^{(n)}(t - 2\pi R)$ \square

We would like to show that $\Phi(\lambda)$ behaves like $\Phi_0(\lambda)$ up to terms of order $R\lambda^n$ or lower. Microlocally, $U(t)$ and $U_S(t)$ behave similarly except on geodesics that enter the neck region that connects the sphere to the rest of X . To isolate these errors we first define pseudodifferential operators on the sphere that microlocalize along such geodesics, and then move them to (X, g_1) . This will allow us to estimate $\Phi_1(\lambda) - \Phi_0(\lambda)$.

Let $\tilde{B} \in \Psi_{\text{phg}}^0(\mathbb{S}^{n+1}(R))$ be chosen so that

$$\text{S-ES}(\tilde{B}) \subset \{(x, \xi) \in S^*\mathbb{S}^{n+1}(R) : \Pi\gamma_t(x, \xi) \in B_{\mathbb{S}^{n+1}(R)}(x_1, 3) \exists t \in \mathbb{R}\},$$

and if $\tilde{A} = I - \tilde{B}$,

$$\text{S-ES}(\tilde{A}) \subset \{(x, \xi) \in S^*\mathbb{S}^{n+1}(R) : \Pi\gamma_t(x, \xi) \notin B_{\mathbb{S}^{n+1}(R)}(x_1, 11/4) \forall t \in \mathbb{R}\}.$$

Note that, here, $\gamma_t(x, \xi)$ is a geodesic on the sphere. By adding smoothing operators if needed, we further require that:

- $\tilde{A}f = 0$ for all $f \in L^2(\mathbb{S}^{n+1}(R))$ with support in $B_{\mathbb{S}^{n+1}(R)}(x_1, 5/2)$, and
- $\text{supp}(\tilde{A}g)$ is contained $\mathbb{S}^{n+1}(R) \setminus B_{\mathbb{S}^{n+1}(R)}(x_1, 5/2)$ for all $g \in L^2(\mathbb{S}^{n+1}(R))$.

Next, we define pseudodifferential operators on X_R as follows. Let $\psi_1 \in \mathcal{C}^\infty(\mathbb{S}^{n+1})$ with

$$(4.14) \quad \psi_1(x) = \begin{cases} 1 & \text{dist}(x, x_1) > 5/2, \\ 0 & \text{dist}(x, x_1) < 9/4, \end{cases}$$

and extend by zero to a smooth, compactly supported function on X_R which we continue to denote by ψ_1 . We then define

$$\begin{aligned} A &= \tilde{A}\psi_1, \\ B &= I - A. \end{aligned}$$

Thus A microlocalizes in S^*X_R to trajectories that enter the gluing region at some time, and B microlocalizes to those that do not.

We now write

$$\begin{aligned} \text{Tr}(U(t)\psi) &= \text{Tr}(U_S(t)) \\ &+ \left[\text{Tr}(U(t)A\psi) - \text{Tr}(U_S(t)\tilde{A}) \right] \\ &- \text{Tr}(U_S(t)\tilde{B}) \\ &+ \text{Tr}(U(t)B\psi) \\ &= T_0(t) + T_1(t) + T_2(t) + T_3(t) \end{aligned}$$

and we will set

$$\Phi_i(\lambda) = \int \varphi_{\varepsilon, 2\pi R}(t) [T_i(t)] dt$$

for $i = 0, 1, 2, 3$. Note that traces involving $U(t)$ are taken in $\mathcal{H}(X_R)$ while those involving $U_S(t)$ are taken in $\mathcal{H}(\mathbb{S}^{n+1}(R))$.

To see that $\Phi_2(\lambda)$ and $\Phi_3(\lambda)$ give $\mathcal{O}(R\lambda^n)$ contributions we need a phase space estimate.

Lemma 4.6. *The estimate*

$$(4.15) \quad \text{vol}_{S^*\mathbb{S}^{n+1}(R)} \left(\text{S-ES}(\tilde{B}) \right) = \mathcal{O}(R)$$

holds.

Proof. Suppose that $\gamma_t(x, \xi)$ enters the cap $B_{\mathbb{S}^{n+1}(R)}(x_1, 3)$ at some time $t \in \mathbb{R}$. Since the geodesic flow has unit speed and the closed geodesics have length $2\pi R$, it will enter first at a time $t \in [0, 2\pi R]$. The volume of the cap $B_{\mathbb{S}^{n+1}(R)}(x_1, 3)$ is of order one. Since phase space volume is preserved by geodesic flow, the phase space volume of points entering the cap, and hence of $\text{S-ES}(\tilde{B})$, is of order $\mathcal{O}(R)$. \square

Remark 4.7. The same estimate holds true for $\text{vol}_{S^*X_R}(\text{S-ES}(B\psi))$ by construction.

Combining Lemma 4.6, Remark 4.7, and Lemma A.1, we immediately obtain:

Lemma 4.8. *The estimate*

$$\Phi_2(\lambda) + \Phi_3(\lambda) = \mathcal{O}(R\lambda^n)$$

holds.

Finally, we prove:

Lemma 4.9. *The estimate $\Phi_1(\lambda) = \mathcal{O}(\lambda^{-\infty})$ holds.*

Proof. First, by the definitions (4.6) and (4.14) of ψ_1 and ψ , it follows that $U(t)A\psi = U(t)A$. Next, note that

- (i) if $\tilde{f} \in L^2(X_R)$ and $\text{supp } \tilde{f} \subset X_R \setminus (S^{n+1} \setminus B_{\mathbb{S}^{n+1}(R)}(x_1, 5/2))$, we have $\tilde{A}\tilde{f} = 0$, and,
- (ii) if $f \in L^2(S^{n+1})$ and $\text{supp } f \subset S^{n+1}(R) \setminus B_{\mathbb{S}^{n+1}(R)}(x_1, 5/2)$, f has a natural identification with $\tilde{f} \in L^2(X_R)$ and

$$Af = \tilde{A}\tilde{f}.$$

It follows that $\text{Tr}(U_S(t)\tilde{A}) = \text{Tr}(\psi_1 U_S(t)\tilde{A})$ and similarly $\text{Tr}(U(t)A) = \text{Tr}(\psi_1 U(t)A)$. Moreover,

$$\text{Tr}_{\mathcal{H}(S^{n+1}(R))}(\psi U_S(t)\tilde{A}) = \text{Tr}_{\mathcal{H}(S^{n+1}(R))}(\psi U_S(t)A)$$

if we regard $U_S(t)$ as acting on the image of $L^2(X_R)$ under A . Hence $T_1(t) = \text{Tr } G_3(t)$ where

$$G_3(t) = \psi_1 U(t)A - \psi_1 U_S(t)A.$$

It suffices to show that $G_3(t)$ is a smoothing operator for all t . We have $G_3(0) = 0$, while

$$(\partial_t - Q)G_3(t) = F_3(t)$$

(recall Q is the generator of $U(t)$) where

$$(4.16) \quad F_3(t) = [\psi_1, Q]U(t)A - [\psi_1, Q]U_S(t)A$$

since the generators of $U(t)$ and $U_S(t)$ coincide in the support of ψ_1 . Since, then

$$(4.17) \quad G_3(t) = \int_0^t U(t-s)F_3(s) ds,$$

it is enough to show that the two right-hand terms in (4.16) are smoothing operators. By propagation of singularities, the operators $\eta U(t)A$ and $\eta U_S(t)A$ are

smoothing for any $\eta \in C_0^\infty(X_R)$ vanishing for x with $\text{dist}(x, x_1) \geq 11/4$. Since the commutators $[Q, \psi_1]$ and $[Q_S, \psi_1]$ are supported in $\{x : 9/4 < \text{dist}(x, x_1) < 5/2\}$, it follows that $F_3(t)$ is smoothing for each t , and hence, by (4.17), $G_3(t)$ is a smoothing operator. \square

Collecting Lemmas 4.8, 4.9, and 4.5, we conclude:

Proposition 4.10. *The asymptotic formula*

$$(4.18) \quad \Phi(\lambda) = c_n R^n \lambda^n + \mathcal{O}_{\varepsilon, \psi}(R\lambda^n)$$

holds.

Proof of Theorem 4.1. Let \mathcal{R}_1 be the set of resolvent resonances for the metric g_1 , and let $u_1(t)$ be the distribution defined in (4.3). The bound (4.2) for the distribution u_0 and the asymptotic formula (4.18) imply that for R sufficiently large and some strictly positive constant b ,

$$|\widehat{\varphi_{\varepsilon, 2\pi R} u_1}(\lambda)| \geq (b - o(1)) \lambda^n$$

as $\lambda \rightarrow +\infty$. We now apply Theorem 4.2 to obtain the conclusion. \square

5. GENERIC LOWER BOUNDS

We fix a compact region $K \subset X$ and we assume that the metric on $X \setminus K'$ is hyperbolic for some compact region $K' \subset X$ containing K . Our goal is to prove that there is a dense G_δ set $\mathcal{M}(g_0, K) \subset \mathcal{G}(g_0, K)$ of metric perturbations for which $N_g(r)$, the resolvent resonance counting function for the perturbed metric has maximal order of growth $n + 1$. By the explicit construction in section 4, the set $\mathcal{M}(g_0, K)$ is nonempty. We follow the ideas of [6] and present the main lines of the argument here. We refer to [4] and [6] for the proofs of statements below that hold with only minor modification in the present context.

5.1. Nevanlinna characteristic functions. We recall briefly the main ideas of [6]. Let f be a function meromorphic of \mathbb{C} . For $r \geq 0$, let $n(r, f)$ be the number of poles of f , including multiplicity, in the region $\{s \in \mathbb{C} : |s - n/2| \leq r\}$. We define an integrated counting function

$$(5.1) \quad N(r, f) \equiv \int_0^r [n(t, f) - n(0, f)] \frac{dt}{t} + n(0, f) \log r.$$

We also need an average of $\log^+ |f|$ along the contour $|s - n/2| = r$:

$$(5.2) \quad m(r, f) \equiv \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(n/2 + re^{i\theta})| d\theta,$$

where $\log^+(a) = \max(0, \log a)$, for $a > 0$. The *Nevanlinna characteristic function*² of f is defined by

$$(5.3) \quad T(r, f) \equiv N(r, f) + m(r, f).$$

²Strictly speaking, this is the Nevanlinna characteristic function of $f(s + n/2)$, rather than that for f . We have chosen to make this minor adaptation here to suit the importance of $s = n/2$ in our parameterization of the spectrum.

This is a nondecreasing function of r . The *order* of a nondecreasing, nonnegative function $h(r) > 0$ is given by

$$(5.4) \quad \limsup_{r \rightarrow \infty} \frac{\log h(r)}{\log r} = \mu,$$

provided it is finite. The order of a meromorphic function f is the order of its characteristic function $T(r, f)$.

The following proposition gives a connection between the order of the characteristic function of f and the order of the pole counting function $n(r, f)$ for f under certain conditions on the meromorphic function f . We recall this result from [6, Lemma 2.3] (see also [4, Lemma 4.2]) with minor changes to suit the convention that the right half-plane $\Re(s) > n/2$ corresponds to the physical region.

Proposition 5.1. *Suppose that $f(s)$ is a meromorphic function on \mathbb{C} with the property that s_0 is a pole of f if and only if $n - s_0$ is a zero of f , and the multiplicities are the same. Furthermore, suppose that no zeros of f lie on the line $\Re(s) = n/2$ and that*

$$(5.5) \quad \int_0^r \frac{d}{dt} \log f(n/2 + it) dt = \mathcal{O}(r^m),$$

for some $m > 1$. Then, f is of order $p > m$ if and only if $n(r, f)$ is of order p .

We next introduce the auxiliary parameter z taking values in an open connected set $\Omega \subset \mathbb{C}$. We consider functions $f(z, s)$ that are meromorphic on $\Omega_z \times \mathbb{C}_s$. Considering $z \in \Omega$ as a parameter, we write $T(z, r, f) \equiv T(r, f(z, \cdot))$ for the Nevanlinna characteristic function of $f(z, s)$.

For any $z_0 \in \Omega$, let $\Omega_0 \subset \Omega$ denote an open ball centered at z_0 . Given $z_0 \in \Omega_0$, there are holomorphic, relatively prime functions g_{Ω_0} and h_{Ω_0} defined on $\Omega_0 \times \mathbb{C}$, so that

$$(5.6) \quad f(z, s) = \frac{g_{\Omega_0}(z, s)}{h_{\Omega_0}(z, s)}, \text{ for } (z, s) \in \Omega_0 \times \mathbb{C}.$$

We suppose that $h_{\Omega_0}(z, s) = (s - n/2)^j \tilde{h}_{\Omega_0}(z, s)$ so that \tilde{h}_{Ω_0} is holomorphic on $\Omega_0 \times \mathbb{C}$ and $\tilde{h}_0(z, n/2)$ is not identically zero. We define a set K_{f, Ω_0} relative to this decomposition by

$$(5.7) \quad K_{f, \Omega_0} = \{z_1 \in \Omega_0 \mid \tilde{h}_{\Omega_0}(z_1, n/2) = 0 \text{ or } h_{\Omega_0}(z_1, s) \text{ vanishes identically, } s \in \mathbb{C}\}.$$

The set K_{f, Ω_0} is independent of the decomposition described above provided each pair $(g_{\Omega_0}, h_{\Omega_0})$ satisfies the same properties. We let K_f be the union of all these sets over balls Ω_0 for each $z_0 \in \Omega$. The intersection of K_f with any compact subset of Ω consists of a finite number of points.

The next result illustrates the utility of the additional parameter z . If the order of the monotone nondecreasing function $r \mapsto T(z, r, f)$ is bounded and the bound is obtained at some $z_0 \in \Omega \setminus K_f$ then it is obtained at all points $z \in \Omega \setminus K_f$ except for a pluripolar set. For the definition of pluripolar sets and additional facts about them see, for example [26] or [23]. Pluripolar sets are small. In particular, we shall use the fact that if $\Omega \subset \mathbb{C}$ is open and $E \subset \Omega$ is pluripolar, then $\Omega \cap \mathbb{R}$ has Lebesgue measure zero.

Theorem 5.2. [6, Theorem 3.5] *Let $\Omega \subset \mathbb{C}$ be an open connected set. Let $f(z, s)$ be meromorphic on $\Omega_z \times \mathbb{C}_s$. Suppose that the order $\rho(z)$ of the function $r \mapsto$*

$T(r, f(z, \cdot))$ is at most ρ_0 for $z \in \Omega \setminus K_f$, and that there is a point $z_0 \in \Omega \setminus K_f$ such that $\rho(z_0) = \rho_0$. Then, there exists a pluripolar set $E \subset \Omega \setminus K_f$ such that $\rho(z) = \rho_0$ for all $z \in \Omega \setminus (E \cup K_f)$.

It follows by Proposition 5.1 that the order of the pole counting function for f , $n(r, f(z, \cdot))$, is the same order ρ_0 for $z \in \Omega \setminus (E \cup K_f)$ provided condition (5.5) and the other hypotheses are satisfied.

5.2. Density of $\mathcal{M}(g_0, K)$. In this subsection we prove

Proposition 5.3. *The set $\mathcal{M}(g_0, K) \subset \mathcal{G}(g_0, K)$ is dense in the C^∞ topology.*

To do this, we need to show that given a metric $\tilde{g} \in \mathcal{G}(g_0, K)$ there is a sequence of metrics in $\mathcal{M}(g_0, K)$ approaching \tilde{g} in the C^∞ topology. If $\tilde{g} \in \mathcal{M}(g_0, K)$, we are, of course, done. If not, noting that $\mathcal{M}(g_0, K) = \mathcal{M}(\tilde{g}, K)$ and $\mathcal{G}(g_0, K) = \mathcal{G}(\tilde{g}, K)$, we may (by relabeling) reduce the problem to assuming that g_0 itself is resonance-deficient, and finding a sequence of metrics in $\mathcal{M}(g_0, K)$ approaching g_0 . In what follows let

$$\sigma_{g, g_0}(s) = \det[S_g(s)S_{g_0}^{-1}(s)].$$

As in section 2 we consider a complex interpolation between a smooth metric g_0 that is hyperbolic outside a compact $K' \subset X$, and a metric $g_1 \in \mathcal{M}(g_0, K)$. The existence of such a metric g_1 is precisely the result of section 4. As in (2.2), this interpolated “metric” is given by $g_z = (1 - z)g_0 + zg_1$, where $z \in \Omega_\epsilon$ with Ω_ϵ as in (2.1).

The scattering matrix $S_{g_z}(s)$ is defined in section 2 along with the corresponding relative scattering phase. We define a relative volume factor (see Corollary 2.6) by

$$\begin{aligned} (5.8) \quad V_{\text{rel}}(z) &\equiv \Delta \text{Vol}(g_z, g_0) \\ &= \int_K (\sqrt{\det(g_z)} - \sqrt{\det(g_0)}) \\ &= \text{Vol}(K, g_z) - \text{Vol}(K, g_0), \end{aligned}$$

and note that this is analytic in z in a possibly smaller region that we still call Ω_ϵ . With c_n the constant from Corollary 2.6, we shall use the function

$$(5.9) \quad f(z, s) = e^{-c_n V_{\text{rel}}(z)(-is)^{n+1}} \sigma_{g_z, g_0}(s),$$

meromorphic in $(z, s) \in \Omega_\epsilon \times \mathbb{C}$.

First, we note from Proposition 2.4 that if s_0 is a pole of f then $n - s_0$ is a zero of f and the multiplicities coincide. Second, using Corollary 2.6, we find that for $a \in \mathbb{R}$ and $t \rightarrow \infty$,

$$(5.10) \quad \begin{aligned} \log f(a, n/2 + it) &= \log \sigma_{g_a, g_0}(n/2 + it) - c_n V_{\text{rel}}(a)t^{n+1} + O(t^n) \\ &= O(t^n). \end{aligned}$$

Consequently, hypothesis (5.5) is

$$(5.11) \quad \int_0^r \frac{d}{dt} \log f(n/2 + it) dt = O(r^n).$$

Hence, from Proposition 5.1, if can can prove that $f(z, s)$ is order $n + 1$ for a large set of $z \in \Omega_z$, it will follow that the corresponding resonance counting function is order $n + 1$ for the same set of z .

To this end, we appeal to Theorem 5.2. We know from section 4 that $f(1, s)$ has the correct order of growth $n + 1$. Furthermore, we note the following bound, which follows directly from Proposition 2.4.

Lemma 5.4. *The order of the function $s \mapsto f(z, s)$ is at most $n + 1$ for $z \in \Omega_\epsilon \setminus K_f$.*

To apply Theorem 5.2 we need, in addition, that $z = 1$ is not in K_f . This may, in fact, fail. But if $1 \in K_f$, we may consider instead the function $f_1(z, s) = f(z, s + i)$. Then $z = 1$ is not in K_{f_1} , because $n/2 + i$ is not a pole of $R_{g_1}(s)$. Thus we may first apply Theorem 5.2 to f_1 , and then apply Proposition 5.1 to f , noting that $s \mapsto f_1(z, s)$ and $s \mapsto f(z, s)$ have the same order. From Theorem 5.2, there exists a pluripolar set $E \subset \Omega$ so that for all $z \in \Omega_\epsilon \setminus (K_f \cup E)$, the resonance counting function has optimal order of growth. Since $(K_f \cup E) \cap \mathbb{R}$ has Lebesgue measure 0, there is a sequence of real $\lambda_j \downarrow 0$ so that $N_{g_{\lambda_j}}(r)$ has maximal order of growth. Then, for any $\epsilon > 0$ there is a $J(\epsilon)$ so that the metric g_{λ_j} satisfies $d_\infty(g_{\lambda_j}, g_0) < \epsilon$ whenever $j > J(\epsilon)$. This finishes the proof of Proposition 5.3.

5.3. The G_δ -Property of $\mathcal{M}(g_0, K)$. The main result of this subsection is:

Proposition 5.5. *The set $\mathcal{M}(g_0, K) \subset \mathcal{G}(g_0, K)$ is a G_δ set.*

If $\mathcal{M}(g_0, K) = \mathcal{G}(g_0, K)$, meaning there are no resonance-deficient metrics in $\mathcal{G}(g_0, K)$, then there is nothing to prove. So suppose there is a resonance-deficient metric $g \in \mathcal{G}(g_0, K)$. Since $\mathcal{M}(g_0, K) = \mathcal{M}(g, K)$, and $\mathcal{G}(g_0, K) = \mathcal{G}(g, K)$, we may, as before, assume g_0 itself is resonance-deficient.

Define, for any $g \in \mathcal{G}(g_0, K)$, $r > 0$,

$$h_g(r) = \frac{1}{2\pi i} \int_0^r t^{-1} \int_{-t}^t \frac{\sigma'_{g, g_0}(n/2 + i\tau)}{\sigma_{g, g_0}(n/2 + i\tau)} d\tau dt + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \log |\sigma_{g, g_0}(n/2 + re^{i\theta})| d\theta.$$

This function is useful because of the following

Lemma 5.6. *If $\limsup_{r \rightarrow \infty} \frac{\log N_{g_0}(r)}{\log r} = p < n + 1$, and $p' > p$, then*

$$\limsup_{r \rightarrow \infty} \frac{\log[\max(h_g(r), 1)]}{\log r} = p'$$

if and only if $N_g(r)$ has order p' .

Proof. Let f be meromorphic in a neighborhood of the closed half plane $\{s : \Re(s) \geq n/2\}$, and such that f has neither zeros nor poles on the line $\Re(s) = n/2$. Let $Z_f(r) = \int_0^r t^{-1} n_{f, Z}(t) dt$ where $n_{f, Z}(r)$ is the number of zeros of $f(s)$ in $\{s : \Re(s) > n/2, |s - n/2| \leq r\}$, and define $P_{f, r}$ analogously as counting the poles of f in the same region. Then

$$(5.12) \quad Z_f(r) - P_f(r) = \frac{1}{2\pi} \Im \int_0^r t^{-1} \int_{-t}^t \frac{f'(n/2 + i\tau)}{f(n/2 + i\tau)} d\tau dt + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \log |f(n/2 + re^{i\theta})| d\theta.$$

This identity follows essentially exactly as the proof of [10, Lemma 6.1], the primary difference being the application of the argument principle for meromorphic, rather than holomorphic, functions.

For $\Re(s_0) > n/2$, if s_0 is a pole of order k of $\sigma_{g,g_0}(s)$, set $\mu_{\text{rel}}(s_0) = -k$; otherwise, set $\mu_{\text{rel}}(s_0)$ to be the order of the zero of $\sigma_{g,g_0}(s)$ at s_0 (of course, $\mu_{\text{rel}}(s_0) = 0$ if s_0 is neither a zero nor a pole). Now we use again, as follows from Proposition 2.4 that for $\Re(s) > n/2$,

$$(5.13) \quad \mu_{\text{rel}}(s) = m_g(n-s) - m_g(s) - m_{g_0}(n-s) + m_{g_0}(s)$$

where m_g (resp., m_{g_0}) is as defined in (1.3) for the metric g (resp. g_0).

In the notation of (5.12), the order of $P_{\sigma_{g,g_0}}(r)$ is at most p , the order of the resonance counting function for Δ_{g_0} . Thus, using (5.12),

$$\limsup_{r \rightarrow \infty} \left(\frac{\log[\max(h_g(r), 1)]}{\log r} \right) = p' > p$$

if and only if the order of $Z_{\sigma_{g,g_0}}(r)$ is p' . The order of $Z_{\sigma_{g,g_0}}(r)$ is the same as the order of $n_{\sigma_{g,g_0},Z}(r)$. Using (5.13) and the fact that $N_{g_0}(r)$ has order p , the order of $n_{\sigma_{g,g_0},Z}(r)$ is $p' > p$ if and only if the order of $N_g(r)$ is p' . \square

Define, for $M, q, j, \alpha > 0$, the set

$$A(M, q, j, \alpha) = \{g \in \mathcal{G}(g_0, K) : \sum_{i,l} g^{il} \xi_i \xi_l \geq \alpha |\xi|^2 \text{ on } K, h_g(r) \leq M(1+r^q) \text{ for } 0 \leq r \leq j\}.$$

Lemma 5.7. *For $M, q, j, \alpha > 0$, the set $A(M, q, j, \alpha)$ is closed.*

Proof. Let $g_m \in A(M, q, j, \alpha)$ be a sequence of metrics converging in the C^∞ topology. Since $\sum_{i,j} g_m^{ij} \xi_i \xi_j \geq \alpha |\xi|^2$, $\{g_m\}$ converges to a metric g with the same property.

Since $g_m \rightarrow g$ in the C^∞ topology, we also have convergence of the cut-off resolvents: for $\chi \in C_c^\infty(X)$, $\chi R_{g_m}(s) \chi \rightarrow \chi R_g(s) \chi$ for values of s for which $\chi R_g(s) \chi$ is a bounded operator. This includes the closed half plane $\{\Re(s) \geq n/2\}$ with the possible exception of a finite number of points corresponding to the discrete spectrum. Thus using the equations (2.7) and (2.8) for the scattering matrix, we see that if $\Re(s_0) \geq n/2$, S_{g_0} has no null space at s_0 , $\Re(s_0) > n/2$, and $s_0(n-s_0)$ is not an eigenvalue of Δ_g , then $S_{g_m}(s) S_{g_0}^{-1}(s_0) \rightarrow S_g(s_0) S_{g_0}^{-1}(s_0)$ in the trace class norm. This convergence is uniform on compact sets which include no poles of either $S_{g_0}^{-1}$ or of R_g . Thus, if the set $\{s : \Re(s) > n/2, |s - n/2| = r\}$ contains no zeros of $S_{g_0}(s)$ or of $S_g(s)$, then $h_{g_m}(r) \rightarrow h_g(r)$. Thus $h_{g_m}(r) \rightarrow h_g(r)$ for all but a discrete set of values of r in $[0, j]$. Since $h_g(r)$ and $h_{g_m}(r)$ are continuous, we get the desired upper bound on $h_g(r)$ for all $r \in [0, j]$. \square

Now, for $M, q, \alpha > 0$, set

$$B(M, q, \alpha) = \bigcap_{j \in \mathbb{N}} A(M, q, j, \alpha).$$

The set $B(M, q, \alpha)$ is closed since $A(M, q, j, \alpha)$ is closed. The proof of Proposition 5.5 is completed by the following lemma.

Lemma 5.8. *If g_0 is resonance-deficient, then*

$$\mathcal{G}(g_0, K) \setminus \mathcal{M}(g_0, K) = \bigcup_{(M,l,m) \in \mathbb{N}^3} B(M, n+1-1/l, 1/m).$$

Proof. If $g \in B(M, n + 1 - 1/l, 1/m)$ for some $M, l, m > 0$, then by Lemma 5.6 the order of growth of $N_g(r)$ is at most the maximum of $n + 1 - 1/l$ and the order of growth of the resonance counting function of N_{g_0} , so $g \notin \mathcal{M}(g_0, K)$.

Suppose $g \in \mathcal{G}(g_0, K) \setminus \mathcal{M}(g_0, K)$. Then the order of $N_g(r)$ is p' for some $p' < n + 1$. An application of Lemma 5.6 shows that there are integers M and l so that $p' < n + 1 - 1/l < n + 1$ and $g \in B(M, n + 1 - 1/l, \alpha)$ for some $\alpha > 0$ sufficiently small. \square

Proof of part (ii) of Theorem 1.1: This is immediate from Propositions 5.3 and 5.5. \square

APPENDIX A. ESTIMATES FOR THE WAVE TRACE

In this appendix we prove a key lemma, essentially taken from Sjöstrand-Zworski [32], which plays an important role in section 4. To formulate the statement, recall that $\Psi_{\text{phg}}^m(M)$ denotes the polyhomogeneous pseudodifferential operators of order m on M . For $P \in \Psi_{\text{phg}}^m(X)$, we recall that the *essential support* of P , denoted $\text{ES}(P)$, is as follows. For a conic open subset U of T^*M , we say that P has order $-\infty$ on U if $|p(x, \xi)| \leq C_N (1 + |\xi|)^{-N}$ for every N and $(x, \xi) \in U$. The essential support $\text{ES}(P)$ is the smallest conic subset of T^*M on the complement of which P has order $-\infty$ (see for example Taylor [33, Chapter VI, Definition 1.3] for discussion). Note that $\text{ES}(P_1 P_2) \subset \text{ES}(P_1) \cap \text{ES}(P_2)$ by the usual symbol calculus (see for example Taylor [33], §0.10 for further discussion). In particular, if P_1 and P_2 have disjoint essential supports, then $P_1 P_2$ is a smoothing operator.

For a pseudodifferential operator A , we set

$$\text{S-ES}(A) = \text{ES}(A) \cap S^*M.$$

We denote by dist_{S^*M} the distance on S^*M induced by the Riemannian metric on S^*M . Since the essential support is a conic set these two notions are equivalent. One should think of the pseudodifferential operators B and C that occur in Lemma A.1 as smoothed characteristic functions of a small region of S^*X so that the operator C has a wave front set slightly bigger than that of B and $B \sim C^2$; compare [32], pp. 854-855. In what follows, Q is a first-order, self-adjoint, scalar pseudodifferential operator (one should think of $Q = \sqrt{\Delta - n^2/4}$ in the application) and $V(t) = \exp(itQ)$; thus Q here occurs in the diagonalization of the matrices Q that occur in section 4).

Lemma A.1. *Let $Q \in OPS_{1,0}^1(M)$ and let $B \in \Psi_{\text{phg}}^0(M)$. Let $\chi \in C_0^\infty(\mathbb{R})$ with support near $t = 0$, $\chi(0) \neq 0$ and $\widehat{\chi}(t) \geq 0$. Let C be a self-adjoint operator in $\Psi_{\text{phg}}^0(X)$ with $(x, \omega) \notin \text{S-ES}(I - C)$ if $\text{dist}_{S^*M}((x, \omega), \text{S-ES}(B)) \leq 1$ and $(x, \omega) \notin \text{S-ES}(C)$ if $\text{dist}_{S^*M}((x, \xi), \text{S-ES}(B)) \geq 2$. Then*

$$\begin{aligned} & \left| \int e^{-i\lambda t} \chi(t - T) \text{Tr}(V(t)B) dt \right| \\ & \leq c_n \chi(0) \|B\| \left(\int_{S^*X} |c(x, \omega)|^2 dx d\omega \right) \lambda^n + \mathcal{O}_{B,T,\chi}(\lambda^{n-1}). \end{aligned}$$

Proof. Following Sjöstrand and Zworski [32] we set $t = T + s$ and write

$$\begin{aligned} M(\lambda) &:= \int e^{-i\lambda t} \chi(t-T) \operatorname{Tr}(V(t)B) dt \\ &= \int e^{-i\lambda T} e^{-i\lambda s} \chi(s) \operatorname{Tr}(e^{iTQ} e^{isQ} B) dt \\ &= e^{-i\lambda T} \operatorname{Tr}(e^{iTQ} \widehat{\chi}(\lambda - Q)B) \end{aligned}$$

so that

$$|M(\lambda)| \leq \|\widehat{\chi}(\lambda - Q)B\|_{\mathcal{I}_1}$$

where we have used the fact that $\|AB\|_{\mathcal{I}_1} \leq \|A\| \|B\|_{\mathcal{I}_1}$ to eliminate the unitary group e^{iTQ} and reduce to a “small-time” estimate. Here and in what follows, $\|\cdot\|$ denotes the operator norm. For any fixed smoothing operator S , $\|\widehat{\chi}(\lambda - Q)S\| = \mathcal{O}(\lambda^{-\infty})$. From the essential support properties of B and C , it is clear that $B(I - C)$ and $(I - C)B$ are smoothing. Moreover, the operator

$$(I - C) \widehat{\chi}(\lambda - Q)B = \int \chi(s) e^{-i\lambda s} (I - C) e^{isQ} B ds$$

obeys the estimate

$$\|(I - C) \widehat{\chi}(\lambda - Q)B\|_{\mathcal{I}_1} \leq \int |\chi(s)| \|(I - C)B(s)\|_{\mathcal{I}_1} ds$$

where $B(s) := e^{isQ} B e^{-isQ}$ has wave front set disjoint from $\operatorname{S-ES}(I - C)$ for small s owing to the support properties of C , so that the trace-norm under the integral is finite. By continuity $\|(I - C)B(s)\|_{\mathcal{I}_1}$ is bounded for small s so that

$$\|(I - C) \widehat{\chi}(\lambda - Q)B\|_{\mathcal{I}_1} \leq C$$

uniformly in λ . Hence, we may estimate

$$\begin{aligned} |M(\lambda)| &\leq \|(I - C) \widehat{\chi}(\lambda - Q)B\|_{\mathcal{I}_1} + \|C \widehat{\chi}(\lambda - Q) (I - C) B\|_{\mathcal{I}_1} + \|C \widehat{\chi}(\lambda - Q) C B\|_{\mathcal{I}_1} \\ &\leq \|B\| \|C \widehat{\chi}(\lambda - Q) C\|_{\mathcal{I}_1} + \mathcal{O}_{B,T,\chi}(1) \end{aligned}$$

where $\mathcal{O}_{B,T,\chi}(1)$ denotes a constant depending on B , T , and χ but independent of λ . Since $\widehat{\chi}$ is positive and C is self-adjoint, we have

$$\begin{aligned} \|C \widehat{\chi}(\lambda - Q) C\|_{\mathcal{I}_1} &= \operatorname{Tr}(C \widehat{\chi}(\lambda - Q) C) \\ &= \int e^{-i\lambda s} \chi(s) \operatorname{Tr}(C^2 e^{isQ}) ds. \end{aligned}$$

We now use Hörmander’s lemma, Lemma A.2 below, to complete the proof. \square

Let X be a compact connected manifold without boundary. Hörmander’s lemma is the following result and appears as [21, Proposition 29.1.2].

Lemma A.2. *Let $B \in \Psi_{\text{phg}}^0(X, \Omega^{1/2}, \Omega^{1/2})$ with principal symbol b and subprincipal symbol b^s , and let P have principal symbol p and subprincipal symbol p^s . Let $E(t)$ solve $(D_t + P)E(t) = 0$ with $E(0) = I$. Let K be the restriction to the diagonal Δ of the Schwarz kernel of $E(t)B$. Then K is conormal with respect to $\Delta \times \{0\}$ for $|t|$ small and*

$$(A.1) \quad K(t, y) = \int \frac{\partial A(y, \lambda)}{\partial \lambda} e^{-i\lambda t} d\lambda,$$

where

$$(A.2) \quad \begin{aligned} A(y, \lambda) &= (2\pi)^{-n} \int_{p(y, \eta) < \lambda} (b + b^s)(y, \eta) \, d\eta \\ &\quad + \frac{\partial}{\partial \lambda} \int_{p(x, \eta) < \lambda} \left(p^s b + \frac{1}{2} \{b, p\} \right) \, d\eta \\ &\quad + (S^{n-2}) \end{aligned}$$

where S^{n-2} means a symbol of order $n - 2$ in the λ variable.

Note that the second integral has lower order so the dominant term gives the leading singularity. Applying this to our case gives the expected leading behavior.

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