

Dedicated to the memory of our friend Pierre Duclos

Conductivity and the Current-Current Correlation Measure

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ABSTRACT. We review various formulations of conductivity for one-particle Hamiltonians and relate them to the current-current correlation measure. We prove that the current-current correlation measure for random Schrödinger operators has a density at coincident energies provided the energy lies in a localization regime. The density vanishes at such energies and an upper bound on the rate of vanishing is computed. We also relate the current-current correlation measure to the localization length.

1. Current-Current Correlation Measure

Higher-order correlation functions for random Schrödinger operators are essential for an understanding of the transport properties of the system. In this paper, we study the current-current correlation measure $M(dE_1, dE_2)$ that describes the correlations between electron currents and is an essential ingredient in the theory of DC conductivity for disordered systems. The connection between the current-current correlation measure (referred to here as the ccc-measure) and the DC conductivity is expressed through the Kubo formula.

We consider a one-particle random, ergodic Hamiltonian H_ω on $\ell^2(\mathbb{Z}^d)$ or on $L^2(\mathbb{R}^d)$. In general H_ω has the form $H_\omega = \frac{1}{2}(-i\nabla - A_0)^2 + V_{per} + V_\omega$, where A_0 is a background vector potential, V_{per} is a real-valued, periodic function, and V_ω is a real-valued, random potential. We always assume that H_ω is self-adjoint on a domain independent of ω . We restrict H_ω to a box $\Lambda \subset \mathbb{R}^d$ with Dirichlet boundary conditions. The finite-volume *current-current correlation measures* $M_{\alpha,\beta}^{(\Lambda)}$, for $\alpha, \beta = 1, \dots, d$, are the point measures on \mathbb{R}^2 defined as follows. Let $E_m(\Lambda)$ be the eigenvalues of H_Λ with normalized eigenfunctions $\phi_m^{(\Lambda)}$. Let $\nabla_\alpha H_\Lambda \equiv i[H_\Lambda, x_\alpha] = p_\alpha - (A_0)_\alpha$ be the α^{th} -component of the local velocity operator, where $p_\alpha = -i\partial/\partial x_\alpha$ is the momentum. The local velocity operator is independent of the randomness. We denote by $v_{\alpha;i,j}^{(\Lambda)}$ the matrix elements of the α^{th} -component of the velocity operator in the eigenstates $\phi_i^{(\Lambda)}$ and $\phi_j^{(\Lambda)}$. That is, we define

$$v_{\alpha;i,j}^{(\Lambda)} \equiv \langle \phi_i^{(\Lambda)}, \nabla_\alpha H_\Lambda \phi_j^{(\Lambda)} \rangle. \quad (1)$$

We now define the *finite-volume, current-current correlation measures* for subsets $\Delta_j \subset \mathbb{R}$ by

$$M_{\alpha,\beta}^{(\Lambda)}(\Delta_1, \Delta_2) \equiv \frac{1}{|\Lambda|} \sum_{i,j: E_i(\Lambda) \in \Delta_1, E_j(\Lambda) \in \Delta_2} v_{\alpha;i,j}^{(\Lambda)} v_{\beta;j,i}^{(\Lambda)}. \quad (2)$$

These set functions extend to point measures on \mathbb{R}^2 . The measures with $\alpha = \beta$, written $M_\alpha^{(\Lambda)} \equiv M_{\alpha,\alpha}^{(\Lambda)}$, are positive Radon measures on \mathbb{R}^2 . We note that $M_{\alpha,\beta}^{(\Lambda)}(\Delta_1, \Delta_2) = M_{\beta,\alpha}^{(\Lambda)}(\Delta_2, \Delta_1)$. We will denote by $M^{(\Lambda)}$ the positive measure $\sum_{\alpha=1}^d M_{\alpha,\alpha}^{(\Lambda)}$. The existence of the infinite-volume limits as $\Lambda \rightarrow \mathbb{R}^d$ of these measures was proved in [25, 32]. We denote the self-averaged, infinite-volume, positive, Borel ccc-measures by $M_\alpha(dE_1, dE_2)$, $M_{\alpha,\beta}(dE_1, dE_2)$, and $M(dE_1, dE_2)$. We remark that these measures can also be defined directly in the infinite volume using the trace-per-unit volume (see, for example, [25]).

In order to relate the ccc-measures to the conductivity, we need to discuss the Fermi distribution. The Fermi distribution $n_F(E; T)$ at energy E and temperature $T \geq 0$ is defined for $T > 0$ by

$$n_F(E; T) \equiv (1 + e^{(E-E_F)/T})^{-1}, \quad (3)$$

whereas for $T = 0$, it is given by

$$n_F(E; 0) = \chi_{(-\infty, E_F]}(E). \quad (4)$$

The density matrix $n_F(E_F; H_\omega)$ corresponding to the $T = 0$ Fermi distribution is simply the spectral projector for H_ω and the half line $(-\infty, E_F]$. We will sometimes write P_{E_F} for this projector.

The Kubo formula for the AC conductivity at temperature $T > 0$ and frequency $\nu > 0$ is derived in linear response theory [8] (see also [3, 27]) by considering a time-dependent Hamiltonian with an external electric field at frequency $\nu > 0$. This Hamiltonian can be written as $H_\omega + \mathcal{E} \cdot x \cos \nu t$, although it is more convenient to study this operator in another gauge. The conductivity relates the induced current to the electric field $E(x, t) = \mathcal{E} \cdot x \cos \nu t$. A formal calculation of the current to linear order in the electric field strength $|\mathcal{E}|$ is sketched in section 4.1. The resulting Kubo formula for the AC conductivity is:

$$\begin{aligned} \sigma_{\alpha,\beta}^{AC}(\nu, T) &\equiv \lim_{\epsilon \rightarrow 0} \lim_{|\Lambda| \rightarrow \infty} \int_{\mathbb{R}^2} \left[\frac{n_F(E_1; T) - n_F(E_2; T)}{E_1 - E_2} \right] \delta_\epsilon(E_1 - E_2 + \nu) M_{\alpha,\beta}^{(\Lambda)}(dE_1, dE_2) \\ &= - \int_{\mathbb{R}^2} \left[\frac{n_F(E_1; T) - n_F(E_2; T)}{E_1 - E_2} \right] \delta(E_1 - E_2 + \nu) M_{\alpha,\beta}(dE_1, dE_2) \\ &= - \int_{\mathbb{R}} \left[\frac{n_F(E + \nu; T) - n_F(E; T)}{\nu} \right] M_{\alpha,\beta}(dE, dE). \end{aligned} \quad (5)$$

where

$$\delta_\epsilon(s) \equiv \frac{\epsilon}{\pi \epsilon^2 + s^2}. \quad (6)$$

We are interested here in the DC conductivity $\sigma_{\alpha,\beta}^{(DC)}$ at zero temperature. Formally, this is obtained from (5) by taking the limits $\nu \rightarrow 0$ and $T \rightarrow 0$. We first take $\nu \rightarrow 0$ in (5), to obtain

$$\sigma_{\alpha,\beta}(0, T) = - \int_{\mathbb{R}} \frac{\partial n_F(E; T)}{\partial E} M_{\alpha,\beta}(dE, dE). \quad (7)$$

Next, we take $T \rightarrow 0$. This results in a delta function $-\delta(E_F - E)$. In order to evaluate the resulting integral, it is convenient to assume that the infinite-volume current-current correlation measure is absolutely continuous with respect to Lebesgue measure on the diagonal and that it has a locally bounded density $m_{\alpha,\beta}(E, E)$. With this assumption, we obtain from (7),

$$\sigma_{\alpha,\beta}^{(DC)}(E_F) = m_{\alpha,\beta}(E_F, E_F). \quad (8)$$

Consequently, the diagonal behavior of the ccc-measure determines the DC conductivity.

In this note, we investigate the existence of the densities $m_{\alpha,\alpha}(E, E)$ and upper bounds on the rate of vanishing for E in certain energy intervals. By (8), this implies the vanishing of the DC conductivity for energies in these intervals. The vanishing of the DC conductivity for energies in the localization regime has been proved by other methods, see section 4. We also relate the localization length to the conductivity, and to the ccc-measure.

We will consider the diagonal behavior of the ccc-measure at energies in the complete localization regime Ξ^{CL} . The complete localization regime is the energy regime discussed by Germinet and Klein [19] and is characterized by strong dynamical localization. We present precise definitions in the sections below. We will also discuss a possibly larger energy domain characterized by a finite localization length defined in (45).

In order to formulate our results, recall that by the Lebesgue Differentiation Theorem, the positive measure M has a density at the diagonal point (E, E) if for any $\epsilon > 0$, with $I_\epsilon(E) = [E, E + \epsilon]$, we have that the limit

$$\lim_{\epsilon \rightarrow 0} \frac{M(I_\epsilon(E), I_\epsilon(E))}{\epsilon^2} \quad (9)$$

exists and is finite. It is necessarily nonnegative. We say that the resulting density $m(E, E)$ vanishes on the diagonal at the point (E, E) at a rate given by a function $g \geq 0$, with $g(s=0) = 0$, if, in addition, for all $\epsilon > 0$ small, we have

$$0 \leq \frac{M(I_\epsilon(E), I_\epsilon(E))}{\epsilon^2} \leq g(\epsilon). \quad (10)$$

The boundedness of the current-current correlation measure on the diagonal means that the measure has no atoms on the diagonal. For comparison, the ccc-measure for the free Hamiltonian is easy to compute due to the fact that the velocity components $\nabla_\alpha H$ commute with the Hamiltonian. A simple calculation shows that

$$M_{\alpha,\beta}(dE_1, dE_2) = n_0(E_1)\delta(E_1 - E_2)\delta_{\alpha,\beta}dE_1dE_2, \quad (11)$$

where $n_0(E)$ is the density of states of the free Laplacian at energy E . Consequently, the limit on the diagonal, as described in (9) does not exist and the DC conductivity is infinite at all energies. Boundedness may be obtained simply for energies in a strong localization regime where (22) holds as stated in Theorem 1. If, in addition, a Wegner estimate holds, then the DC conductivity vanishes.

We now state our main result on the vanishing of the ccc-density on the diagonal.

THEOREM 1. *Let $E \in \Xi^{CL}$. Then, the current-current correlation measure density $m(E, E)$ exists and vanishes on the diagonal. If, in addition, the Wegner estimate (39) holds on Ξ^{CL} , then for any bounded interval $I_0 \subset \Xi^{CL}$ with $E \in I_0$, and for any $0 < s < 1$, there is a finite constant $C_{I_0, s} < \infty$ so that*

$$\frac{1}{\epsilon^2} M(I_\epsilon(E), I_\epsilon(E)) \leq C_{I_0, s} \epsilon |\log \epsilon|^{\frac{1}{s}}, \quad (12)$$

where $I_\epsilon(E) = [E, E + \epsilon]$.

We note that if, in addition to the Wegner estimate and the other hypotheses of Theorem 1, we know that Minami's estimate is satisfied in a neighborhood of a given energy, then the rate of vanishing can be improved for certain intervals near the diagonal (see Corollary 1). We discuss this and some generalizations in section 2 after the proof of the main theorem.

REMARK 1. Suppose that E_0 is a lower band edge at which the IDS exhibits a Lifshitz tail behavior. For example, if E_0 is a band edge of the deterministic spectrum Σ and $E > E_0$, the existence of Lifshitz tails [23, 28] means that for E sufficiently close to E_0 , the IDS $N(E)$ satisfies

$$N(E) - N(E_0) \leq C_{E_0} e^{-\frac{\alpha}{(E-E_0)^{d/2}}}. \quad (13)$$

We write $E(I_\epsilon(E_0))$ for the spectral projector for H_ω and the interval $I_\epsilon(E_0)$. It follows from (13) that for $\epsilon > 0$ small enough, we have

$$\mathbb{E}\{\text{Tr} \chi_0 E(I_\epsilon(E_0)) \chi_0\} \leq C_{E_0} e^{-\frac{\alpha}{\epsilon^{d/2}}}, \quad (14)$$

for some constant $\alpha > 0$ and where χ_0 is the characteristic function for the unit cube about zero. We use this estimate in (40) below in place of the usual Wegner estimate (39). As a consequence, we obtain an exponential rate of vanishing on the diagonal. There is a finite, constant $D_0 > 0$ so that for all $\epsilon > 0$ small enough,

$$\frac{1}{\epsilon^2} M(I_\epsilon(E_0), I_\epsilon(E_0)) \leq D_0 e^{-\alpha/\epsilon^{d/2}}. \quad (15)$$

Of course, this only occurs at a countable set of energies (the lower band edges). This result, however, stating that, roughly, $m(E + \epsilon, E)$ vanishes at an exponentially fast rate as $\epsilon \rightarrow 0$, is consistent with the Mott theory of conductivity.

REMARK 2. The optimal rate of vanishing in Theorem 1 is not known. We note, however, the relationship between the localization length (see section 3) and the ccc-measure [5, section V]:

$$\ell^2(\Delta) = 2 \int_{\Delta \times \mathbb{R}} \frac{M(dE_1, dE_2)}{(E_1 - E_2)^2}, \quad (16)$$

from which it follows that the ccc-density must vanish faster than $\mathcal{O}(\epsilon)$ on the diagonal if the localization length is to be finite. We provide a proof of (16) as part of Proposition 1 in section 3.

This leads us to our second result relating the localization length $\ell(\Delta)$, for an interval $\Delta \subset \mathbb{R}$, to the ccc-measure. The localization length bounds the ccc-measure and the vanishing of the localization length implies the vanishing of

the DC conductivity. For this result, we do not need the hypothesis of complete localization.

THEOREM 2. *For $E \in \mathbb{R}$, and for any $\epsilon > 0$ small, define the interval $I_\epsilon(E) \equiv [E, E + \epsilon]$. Suppose that there is a constant $0 \leq M_E < \infty$ so that the localization length $\ell(I_\epsilon(E)) \leq M_E < \infty$, for all $\epsilon > 0$ small. Then, the density of the ccc-measure $m(E, E)$ exists on the diagonal. If, in addition, $\ell(I_\epsilon(E)) \rightarrow 0$ as $\epsilon \rightarrow 0$, then we have*

$$\lim_{\epsilon \rightarrow 0} \frac{M(I_\epsilon(E), I_\epsilon(E))}{\epsilon^2} = 0. \quad (17)$$

Consequently, $\sigma^{(DC)}(E) = 0$.

Finally, we mention the open question of the existence of a density $m(E_1, E_2)$ for $E_1 \neq E_2$ that we refer to as the off-diagonal case. This is important in view of the formula for the localization length (16), and in order to control the density in a neighborhood of the diagonal. Unfortunately, the approach developed in this note does not seem to allow us to control the off-diagonal behavior. We mention that for lattice models on $\ell^2(\mathbb{Z}^d)$, Bellissard and Hislop [6] proved the existence of a density $m_\alpha(E_1, E_2)$ at off-diagonal energies (E_1, E_2) provided that 1) the energies lie in a region away from the diagonal $E_1 = E_2$ determined by the strength of the disorder, and 2) the density of the single-site probability measure extends to a function analytic in a strip around the real axis.

2. Existence and Vanishing of the Density on the Diagonal in the Complete Localization Regime

In this section, we give a simple proof of Theorem 1 on the existence of, and vanishing of, the ccc-density on the diagonal $m(E, E)$ when $E \in \Xi^{CL}$. We denote by C a generic, nonnegative, finite constant whose value may change from line to line. We work in the infinite-volume framework using the trace-per-unit volume and refer the reader to [8] and [5] for a complete discussion. Let (Ω, \mathbb{P}) be a probability space with a \mathbb{Z}^d -ergodic action $\tau_a : \Omega \rightarrow \Omega$, $a \in \mathbb{Z}^d$. Let $a \in \mathbb{Z}^d \rightarrow U_a$ be a unitary representation of \mathbb{Z}^d on \mathcal{H} . A *covariant operator* A on a separable Hilbert space \mathcal{H} is a \mathbb{P} -measurable, operator-valued function $A = \{A_\omega \mid \omega \in \Omega\}$ such that for $a \in \mathbb{Z}^d$, we have $(U_a A U_{-a})_\omega = A_{\tau_{-a}\omega}$. Let χ_0 be the characteristic function on the unit cube centered at the origin. The *trace-per-unit volume* of a covariant operator A , denoted by $\mathcal{T}(A)$, is given by

$$\mathcal{T}(A) \equiv \mathbb{E}\{\text{Tr}(\chi_0 A \chi_0)\}, \quad (18)$$

when it is finite.

We recall that for $\Delta_j \subset \mathbb{R}$, with $j = 1, 2$, the positive ccc-measure M_α is given by

$$\begin{aligned} M_\alpha(\Delta_1, \Delta_2) &= \mathcal{T}(\nabla_\alpha H E(\Delta_1) \nabla_\alpha H E(\Delta_2)) \\ &= \lim_{|\Lambda| \rightarrow \infty} \frac{1}{|\Lambda|} \text{Tr} \chi_\Lambda \nabla_\alpha H E(\Delta_1) \nabla_\alpha H E(\Delta_2) \chi_\Lambda. \end{aligned} \quad (19)$$

We recall from [8, section 3.2] that \mathcal{K}_2 is the space of all measurable, covariant operators so that

$$\mathbb{E}\{\|A\chi_0\|_2^2\} < \infty. \quad (20)$$

We refer to [8] for further definitions and details. Because the spectral projector $E(\Delta) \in \mathcal{K}_2$, the cyclicity of \mathcal{T} permits us to write

$$\mathcal{T}(\nabla_\alpha H E(\Delta_1) \nabla_\alpha H E(\Delta_2)) = \mathcal{T}(E(\Delta_2) \nabla_\alpha H E(\Delta_1) \nabla_\alpha H E(\Delta_2)). \quad (21)$$

Let χ_x be a bounded function of compact support in a neighborhood of $x \in \mathbb{R}^d$. Let $\|\cdot\|_p$ denote the p^{th} -trace norm for $p \geq 1$. We give a characterization of the region Ξ^{CL} of complete localization given in [18] and [19].

DEFINITION 1. *An energy $E \in \Sigma$ belongs to Ξ^{CL} if there exists a open neighborhood I_E of E such that the following bound holds. Let \mathcal{I}_E denote all functions $f \in L_0^\infty(\mathbb{R})$, with $\text{supp } f \subset I_E$ and $\sup_{t \in \mathbb{R}} |f(t)| \leq 1$. Then, for any $0 < s < 1$, there is a finite constant $C_s > 0$ so that the Hilbert-Schmidt bound holds*

$$\sup_{f \in \mathcal{I}_E} \mathbb{E}\{\|\chi_x f(H_\omega) \chi_y\|_2^2\} \leq C_s e^{-\|x-y\|^s}, \text{ for all } x, y \in \mathbb{R}^d. \quad (22)$$

REMARK 3. This estimate can be improved to $s = 1$ by the method of fractional moments [2, 4].

PROOF OF THEOREM 1. 1. We begin with formula (21) and take $\Delta_1 = \Delta_2 = I_\epsilon(E) = [E, E + \epsilon]$. We write the velocity operator as a commutator

$$\nabla_\alpha H = i[H, x_\alpha] = i[H - E, x_\alpha]. \quad (23)$$

For brevity, we write $P_\epsilon \equiv E(I_\epsilon(E))$. We then have

$$\begin{aligned} M_\alpha(I_\epsilon(E), I_\epsilon(E)) &= \mathcal{T}(P_\epsilon \nabla_\alpha H P_\epsilon \nabla_\alpha H P_\epsilon) \\ &= -\mathcal{T}(P_\epsilon [H - E, x_\alpha] P_\epsilon [H - E, x_\alpha] P_\epsilon). \end{aligned} \quad (24)$$

Next, we expand the commutators and obtain four terms involving P_ϵ . This is facilitated by the introduction of a function $f_\epsilon(s) \equiv (s - E)\chi_{I_\epsilon(E)}(s)/\epsilon$, so that $|f_\epsilon| \leq 1$. We obtain a factor of ϵ from each projector so that

$$M_\alpha(I_\epsilon(E), I_\epsilon(E)) = -\epsilon^2(I + II + III + IV), \quad (25)$$

where we have:

$$I = \mathcal{T}(f_\epsilon(H) x_\alpha f_\epsilon(H) x_\alpha P_\epsilon); \quad (26)$$

$$II = -\mathcal{T}(f_\epsilon(H) x_\alpha P_\epsilon x_\alpha f_\epsilon(H)); \quad (27)$$

$$III = -\mathcal{T}(P_\epsilon x_\alpha f_\epsilon^2(H) x_\alpha P_\epsilon); \quad (28)$$

$$IV = \mathcal{T}(P_\epsilon x_\alpha f_\epsilon(H) x_\alpha f_\epsilon(H)). \quad (29)$$

2. In order to estimate these four terms for $E \in \Xi^{CL}$, we need the following bounds. Let $F_{\alpha, \epsilon}(H_\omega)$ denote the operator $x_\alpha f_\epsilon(H_\omega) x_\alpha$. The bound in Definition 1 implies the following estimate for any integer $q > 0$ and s as in definition 1:

$$\sup_{f \in \mathcal{I}_\epsilon} \mathbb{E}\{\|(F_{\alpha, \epsilon}(H_\omega))^{q/2} f(H_\omega) \chi_0\|_2^2\} \leq C(s, q, d) \Gamma((q + d)/s)^2, \quad (30)$$

where $\Gamma(t)$ is the gamma function and $C(s, q, d)$ depends on s through the constants C_s and α_s as in (22) and grows linearly in q . To prove this bound, we

use a partition of unity on \mathbb{R}^d given by $\sum_{k \in \mathbb{Z}^d} \chi_k = 1$ and we define operators $A_{k\ell} \equiv \chi_k f(H_\omega) \chi_\ell$ and $B_{k\ell} \equiv \chi_k (x_\alpha f(H_\omega) x_\alpha)^q \chi_\ell$. We then estimate the Hilbert-Schmidt norm in (30) by

$$\|(F_{\alpha,\epsilon}(H_\omega))^{q/2} f(H_\omega) \chi_0\|_2^2 = \|\chi_0 f(H_\omega) (F_{\alpha,\epsilon}(H_\omega))^q f(H_\omega) \chi_0\|_1 \quad (31)$$

$$\leq \sum_{m,k \in \mathbb{Z}^d} \|A_{0m}\|_2 \|A_{\ell 0}\|_2 \|B_{m\ell}\|. \quad (32)$$

Since $\|B_{m\ell}\| \leq C(q) \|m\|^q \|\ell\|^q$, where $C(q)$ is linear in q for integer q , we obtain from (22) and (31), and the integral representation of the gamma function

$$\mathbb{E}\{\|(F_{\alpha,\epsilon}(H_\omega))^{q/2} f(H_\omega) \chi_0\|_2^2\} \leq C(s) \left(\sum_{m \in \mathbb{Z}^d} |m|^q e^{-\alpha_s \|m\|^s} \right)^2 \quad (33)$$

$$\leq C(s, q, d) \Gamma((q+d)/s)^2, \quad (34)$$

proving the estimate (30). We will also use the operator bounds

$$\|f(H)\| \leq 1, \quad |f| \leq 1, \quad (35)$$

and by a simple trace class estimate,

$$\mathcal{T}(P_\epsilon) \leq C, \quad (36)$$

for some finite constant $C > 0$.

3. We now bound the term I in (26) as follows, recalling that $|f_\epsilon| \leq 1$ and that $P_\epsilon f_\epsilon(H_\omega) = f_\epsilon(H_\omega)$:

$$\begin{aligned} |I| &\leq |\mathbb{E}\{\text{Tr} \chi_0 f_\epsilon(H) x_\alpha f_\epsilon(H) x_\alpha P_\epsilon \chi_0\}| \\ &\leq \mathcal{T}(P_\epsilon)^{1/2} (\mathbb{E}\{\|F_{\alpha,\epsilon}(H_\omega) f_\epsilon(H_\omega) \chi_0\|_2^2\})^{1/2} \\ &= \mathcal{T}(P_\epsilon)^{1/2} \{\mathcal{T}(f_\epsilon(H_\omega) F_{\alpha,\epsilon}(H_\omega)^2 f_\epsilon(H_\omega) P_\epsilon)\}^{1/2} \\ &\leq \mathcal{T}(P_\epsilon)^{1/2+1/4} (\mathbb{E}\{\|F_{\alpha,\epsilon}(H_\omega)^2 f_\epsilon(H_\omega) \chi_0\|_2^2\})^{1/4} \end{aligned} \quad (37)$$

Bounds (30), together with (35)-(36), show that the term in (37) is uniformly bounded as $\epsilon \rightarrow 0$. It is clear that terms $II - IV$ are bounded in a similar manner. As a consequence, it follows from these bounds and (25) that

$$\lim_{\epsilon \rightarrow 0} \frac{M_\alpha(I_\epsilon(E), I_\epsilon(E))}{\epsilon^2} \leq C < \infty. \quad (38)$$

This implies that the density $m(E, E)$ exists for $E \in \Xi^{CL}$.

4. The rate of vanishing can be calculated with more careful estimates. We use the Wegner estimate in place of (36) in order to obtain more powers of ϵ . The Wegner estimate for infinite-volume operators has the form

$$\mathcal{T}(E(J)) = \mathbb{E}\{\text{Tr} \chi_0 E(J) \chi_0\} \leq C_0 |J|, \quad (39)$$

for any subset $J \subset \mathbb{R}$. Turning to term I in (37), we iterate the argument n times, recalling that $P_\epsilon f_\epsilon(H_\omega) = f_\epsilon(H_\omega)$, and obtain

$$|I| \leq \mathcal{T}(P_\epsilon)^{\frac{1}{2} + \dots + \frac{1}{2^n}} \left(\mathbb{E}\{\|(F_{\alpha,\epsilon}(H_\omega))^{2^{n-1}} f_\epsilon(H_\omega) \chi_0\|_2^2\} \right)^{\frac{1}{2^n}}. \quad (40)$$

We use the strong localization bound (30) in (40), together with the Wegner estimate (39) for $I_\epsilon(E)$, and Stirling's formula for the gamma function, to obtain

$$|I| \leq C(s, d) \epsilon^{1 - \frac{1}{2^n}} (2^n)^{\frac{1}{s}}. \quad (41)$$

We choose n so that $2^n \sim |\log \epsilon|$ producing the upper bound $\epsilon |\log \epsilon|^{\frac{1}{s}}$. The remaining terms can be estimated in a similar manner. \square

REMARK 4. We note that if we consider intervals of the form

$$I_\epsilon = [E, E + \epsilon] \times [E - \epsilon, E] \subset \mathbb{R}^2, \quad (42)$$

we can improve the rate of vanishing. This is a consequence of the analysis in [27] in the context of Mott's formula for the AC conductivity. It requires the Minami estimate in addition to the Wegner estimate. We recall that Minami's estimate for local operators H_ω^Λ is a second-order correlation estimate. We assume that the single-site probability measure has a bounded density ρ . Let $E_\Lambda(\Delta)$ denote the spectral projector for H_ω^Λ and the interval Δ . The Minami estimate is:

$$\mathbb{E} \{ \text{Tr} E_\Lambda(\Delta) (\text{Tr} E_\Lambda(\Delta) - 1) \} \leq (\|\rho\|_\infty |\Delta| |\Lambda|)^2. \quad (43)$$

This was proved by Minami [30] for lattice models on \mathbb{Z}^d , and recent, simplified proofs have appeared in [7, 10, 24]. More recently, the Minami estimate was proved in [11] for certain Anderson models in the continuum for an interval of energy near the bottom of the deterministic spectrum.

COROLLARY 1. *For any energy $E \in \Xi^{CL}$ for which both the Wegner and Minami estimates hold in an interval I_0 containing E , we have*

$$\frac{1}{\epsilon^2} M([E, E + \epsilon], [E - \epsilon, E]) \leq C_{I_0} \epsilon^2 |\log \epsilon|^{d+2}. \quad (44)$$

PROOF. As mentioned, this result is a consequence of the analysis in [27] of the Mott formula. We go back to the decomposition given in (25) of that paper, and estimate each term using [27, Theorem 4.1]. Applying [27, Theorem 4.1] directly to bound $M([E, E + \epsilon], [E - \epsilon, E])$, we obtain (44). \square

3. Relation to localization length

The second moment of the position operator is used in the following covariant definition of the localization length due to Bellissard, van Elst, and Schulz-Baldes [5].

DEFINITION 2. *The localization length for an interval $\Delta \subset \mathbb{R}$ is*

$$\ell(\Delta)^2 \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathcal{T}(E(\Delta) |x(t) - x|^2 E(\Delta)) dt, \quad (45)$$

where $x(t) \equiv e^{-itH} x e^{itH}$.

If $\Delta \subset \mathbb{R}$ is contained in the complete localization region it is easily seen from (22) that the localization length is finite but one expects that the finiteness of the localization length holds in a much larger energy domain. Notice, however, that as shown in [5], the finiteness of $\ell(\Delta)$ implies that the spectrum of H_ω in Δ is pure point. This does not *a priori* imply estimates such as (22).

The ccc-measure on the diagonal and the localization length are very closely related. Recall that the velocity operator is $\nabla H = i[H, x]$.

PROPOSITION 1. *Let $E(d\lambda)$ be the spectral family for H_ω . The ccc-measure is related to the localization length by*

$$\ell(\Delta)^2 = 2 \int_{\mathbb{R}} \int_{\Delta} \frac{\mathcal{T}(E(d\mu)\nabla H E(d\lambda)\nabla H E(d\mu))}{(\lambda - \mu)^2}. \quad (46)$$

Furthermore, we have the bound

$$\frac{\mathcal{T}(E(\Delta)\nabla H E(\Delta)\nabla H E(\Delta))}{|\Delta|^2} \leq \ell(\Delta)^2. \quad (47)$$

PROOF. We use the Fundamental Theorem of Calculus to write

$$x(t) - x = - \int_0^t e^{-isH} V e^{isH} ds. \quad (48)$$

Taking the absolute square, we obtain

$$|x(t) - x|^2 = \int_0^t dw \int_0^t ds e^{-isH} \nabla H e^{i(s-w)H} \nabla H e^{iwH}. \quad (49)$$

Using the spectral family for H in the form $\int_{\mathbb{R}} E(d\lambda) = 1$, we obtain

$$\begin{aligned} & \mathcal{T}(E(\Delta)|x(t) - x|^2 E(\Delta)) \\ &= \int_{\Delta} \int_{\mathbb{R}} \left\{ \int_0^t ds e^{-is(\mu-\lambda)} \int_0^t dw e^{iw(\mu-\lambda)} \mathcal{T}(E(d\mu)\nabla H E(d\lambda)\nabla H E(d\mu)) \right\}. \end{aligned} \quad (50)$$

We first perform the integration over s and w . We next take the time average. Finally, taking the limit $T \rightarrow \infty$, we obtain (46).

In order to obtain (47), we write the projectors in the trace-per-unit volume on the left of (47) as $E(\Delta) = \int_{\Delta} E(d\mu)$ and note that $(\lambda - \mu)^2 \leq |\Delta|^2$ for $\lambda, \mu \in \Delta$. \square

The simple proof of Theorem 2 follows from (46). For $E \in \mathbb{R}$, and $\epsilon > 0$ small, we have

$$\begin{aligned} \frac{2}{\epsilon^2} M(I_\epsilon(E), I_\epsilon(E)) &= \frac{2}{\epsilon^2} \int \int_{I_\epsilon(E) \times I_\epsilon(E)} \mathcal{T}(E(d\mu)\nabla H E(d\lambda)\nabla H E(d\mu)) \\ &\leq \ell(I_\epsilon(E))^2. \end{aligned} \quad (51)$$

Taking $E_1 = E_2$, we obtain the proof of the second part of the theorem.

We next present an explicit formula relating the localization length $\ell(\Delta)$ to the second moment of the position operator.

PROPOSITION 2. *We have the following identity:*

$$\frac{1}{2} \ell(\Delta)^2 = \mathcal{T}(E(\Delta)|x|^2 E(\Delta)) - \int_{\Delta} \mathcal{T}(E(d\lambda)x E(d\lambda)x E(d\lambda)) \geq 0. \quad (52)$$

where $E(d\lambda)$ is the spectral family for H_ω . Consequently, if the second moment of the position operator $\mathcal{T}(E(\Delta)|x|^2 E(\Delta))$ on the right side of (52) is finite, then the localization length is finite.

PROOF. The proof of the equality in (52) follows from (46) by expanding the commutator in the definition of ∇H . The nonnegativity is obvious and this proves the second part of the proposition. \square

We remark that the formulas in (45) and in (52) are manifestly covariant with respect to lattice translations. It is known [5] that $\ell(\Delta) < \infty$ implies that the spectrum of H_ω in Δ is pure point almost surely. The finiteness of the term $\mathcal{T}(E(\Delta)|x|^2E(\Delta))$ should be compared with the finiteness condition of the Fermi projector given in (57) for the continuum model and in (65) for the lattice model.

4. Conductivity for Random Schrödinger Operators

In this section, we discuss the various notions of conductivity that have occurred in recent literature and provide a justification for the calculation of the conductivity presented in section 1. As a starting point, we begin with the adiabatic approach to the Kubo formula as presented in [8]. We then consider related results of Fröhlich and Spencer (1983) [17], Kunz (1987) [29], Bellissard, van Elst, and Schulz-Baldes (1994) [5], Aizenman-Graf (1998) [3], and Nakano (2002) [31]. We write v_α or $\nabla_\alpha H$ for the α^{th} -component of the velocity operator $v_\alpha = \nabla_\alpha H = i[H, x_\alpha]$.

We mention another derivation of the Kubo-Streda formula for the transverse conductivity in two-dimensions for perturbations of the Landau Hamiltonian at zero frequency and temperature is presented in [14, section IV]. This derivation does not require the use of an electric field that is adiabatically switched on. On the other hand, the Fermi level has to be restricted to a spectral gap of the Hamiltonian, rather than to the region of localization.

4.1. The Adiabatic Definition of the Conductivity.

4.1.1. *Kubo formula through linear response theory.* In the paper of Bouclet, Germinet, Klein and Schenker [8], the total charge transport is calculated in linear response theory in which the electric field is adiabatically switched on. This provides a rigorous derivation of the DC Kubo formula for the conductivity tensor. A noncommutative integration approach is presented in [16, 15] which somewhat simplifies manipulation of operators. The random, \mathbb{Z}^d -ergodic Hamiltonian H_ω on $L^2(\mathbb{R}^d)$ has the form

$$H_\omega = (i\nabla + A_\omega)^2 + V_\omega, \quad (53)$$

where (A_ω, V_ω) are random variables so that this operator is self-adjoint. We add a time-dependent, homogeneous, electric field $\mathcal{E}(t)$ that is adiabatically switched on from $t = -\infty$ until $t = 0$, when it obtains full strength. Such a field is represented by $\mathcal{E}(t) = e^{\eta t} \mathcal{E}$, where \mathcal{E} is a constant and $t_- \equiv \min(0, t)$. The usual gauge choice is the time-dependent Stark Hamiltonian given by

$$\tilde{H}_\omega(t) = H_\omega + \mathcal{E}(t) \cdot x. \quad (54)$$

This Hamiltonian is not bounded from below. As is well-known, and used in [8], one can make another choice of gauge to eliminate this technical problem.

Let $F(t)$ be the function given by

$$F(t) = \int_{-\infty}^t \mathcal{E}(s) ds, \quad (55)$$

and note that the integral converges provided $\eta > 0$.

We now consider another Hamiltonian obtained from H_ω in (53) by a time-dependent gauge transformation using the operator $G(t) = e^{iF(t) \cdot x}$:

$$H_\omega(t) \equiv G(t)H_\omega G^*(t) = (i\nabla + A_\omega + F(t))^2 + V_\omega. \quad (56)$$

This operator is manifestly bounded from below almost surely provided V_ω is also. Furthermore, the two Hamiltonians $H_\omega(t)$ and $\tilde{H}_\omega(t)$, given in (54), are physically equivalent in that they generate the same dynamics. If ψ_t solves the Schrödinger equation generated by $H_\omega(t)$, then the gauge transformed wave function $G(t)^* \psi_t$ solves the Schrödinger equation associated with $\tilde{H}_\omega(t)$. Consequently, we will work with $H_\omega(t)$.

In order to describe the current at time $t = 0$, the system is prepared in an initial equilibrium state ξ_ω at time $t = -\infty$. The initial state is usually assumed to be the density matrix corresponding to the Fermi distribution at temperature $T \geq 0$. If $T > 0$, the state with Fermi energy E_F is given by $\xi_\omega = n_F(H_\omega; T)$, where $n_F(E; T)$ is given in (3). If $T = 0$, the initial equilibrium state is the Fermi projector $\xi_\omega = n_F(H_\omega; 0) = \chi_{(-\infty, E_F]}(H_\omega)$. We will often write P_{E_F} for this projection. More general states are allowed, see [8, section 5]. The crucial assumption for the $T = 0$ case is

$$\mathbb{E}\{\| |x| P_{E_F} \chi_0 \|_2^2\} < \infty, \quad (57)$$

which holds true whenever the Fermi level lies in an interval of complete localization Ξ^{CL} [20]. If the initial state is given by a Fermi distribution, together with (57) if $T = 0$, the authors prove the existence of a unique time-dependent density matrix $\rho_\omega(t)$ solving the following Cauchy problem:

$$\begin{aligned} i\partial_t \rho_\omega(t) &= [H_\omega(t), \rho_\omega(t)]_* \\ \lim_{t \rightarrow -\infty} \rho_\omega(t) &= \xi_{E_F}, \end{aligned} \quad (58)$$

in suitable noncommutative L^1 and L^2 spaces (see [15, 16] for the construction and use of the more natural L^p spaces defined over the reference von Neumann algebra of the problem, and the corresponding Sobolev spaces). In particular, the commutator in (58) requires some care in its definition when H_ω is an unbounded operator. The subscript star reminds the reader of these noncommutative integration spaces.

In [8], the authors also prove that in the $T = 0$ case, the density matrix $\rho_\omega(t)$ is an orthogonal projection for all time. Given this density matrix $\rho_\omega(t)$, the current at time $t = 0$ is defined to be:

$$J(\eta, \mathcal{E}; E_F) \equiv \mathcal{T}(\tilde{v}(0)\rho_\omega(0)), \quad (59)$$

where the modified velocity operator is $\tilde{v}(0) \equiv i[H_\omega, x] - 2F(0) = v - 2F(0)$. At time $t = -\infty$, the system is assumed to be in equilibrium so the initial current is zero. The current $J(\eta, \mathcal{E}; E_F)$ is the net current at time zero obtained by adiabatically switching on the electric field.

Let $U(s, t)$ be the unitary propagator for the time-dependent Hamiltonian (56). We denote by $\xi_{E_F}(\tau)$ the time-evolved operator obtained from the Fermi distribution $n_F(H_\omega(\tau); T)$, as in (3) for $T > 0$, or as in (4), for $T = 0$. The components of the current are given explicitly by

$$J_\alpha(\eta, \mathcal{E}; E_F) = -\mathcal{T} \left\{ \int_{-\infty}^0 d\tau e^{\eta\tau} \tilde{v}_\alpha(0) (U(0, \tau)(i[\mathcal{E} \cdot x, \xi_{E_F}(\tau)])U(\tau, 0))_* \right\}. \quad (60)$$

Linear response theory now states that the conductivity is the constant of proportionality between the current J and the electric field \mathcal{E} , that is, that Ohm's Law holds: $J = \sigma\mathcal{E}$. This is a tensorial relationship. Formally, one should expand $J(\eta, \mathcal{E}; E_F)$, given in (60), about $\mathcal{E} = 0$. The components $\sigma_{\alpha\beta}$ are obtained via the derivative:

$$\sigma_{\alpha\beta}(\eta, E_F) = \left(\frac{\partial J_\alpha(\eta, \mathcal{E}; E_F)}{\partial \mathcal{E}_\beta} \right)_{\mathcal{E}=0}. \quad (61)$$

The question of the limit $\mathcal{E} \rightarrow 0$ is addressed in the following theorem.

THEOREM 3. [8, Theorem 5.9] *For $\eta > 0$, the current defined in (59) is differentiable with respect to \mathcal{E}_j at $\mathcal{E} = 0$, and the conductivity tensor is given by*

$$\sigma_{\alpha\beta}(\eta; E_F) = -\mathcal{T} \left\{ \int_{-\infty}^0 d\tau e^{\eta\tau} v_\alpha (e^{-i\tau H_\omega}(i[x_\beta, \xi_{E_F}])e^{i\tau H_\omega})_* \right\}. \quad (62)$$

Once again, the subscript star in (62) indicates that operators and product of operators are considered within suitable noncommutative integration spaces. Formula (62) is valid for a large family of equilibrium initial states including the Fermi distributions. The analogue of [5, Eq. (41)] and [34, Theorem 1] then holds.

COROLLARY 2. [8, Corollary 5.10] *Assume that $\mathcal{E}(t) = \text{Re } e^{i\nu t}$, $\nu \in \mathbb{R}$, then the conductivity $\sigma_{jk}(\eta; \zeta; \nu)$ at frequency ν is given by*

$$\sigma_{jk}(\eta; \zeta; \nu; 0) = -\mathcal{T} \{ v_\alpha (i\mathcal{L}_1 + \eta + i\nu)^{-1} (\partial_\beta \zeta) \}, \quad (63)$$

where \mathcal{L}_1 is the Liouvillian generating (58), and $\partial_\beta \zeta = i[x_\beta, \xi_{E_F}]$.

For $T = 0$, following ideas of [5], the authors recover the Kubo-Středa formula for the conductivity tensor. The distribution ξ_{E_F} is given by the Fermi projector as in (4). In this case, we can take the limit $\eta \rightarrow 0$ and obtain an expression for the DC conductivity.

COROLLARY 3. [8, Theorem 5.11] *Under the hypothesis (57) on the Fermi projector P_{E_F} , the conductivity tensor is given by*

$$\sigma_{\alpha,\beta}^{DC}(E_F) = \sigma_{\alpha\beta} = -i\mathcal{T} \{ P_{E_F} [[x_\alpha, P_{E_F}], [x_\beta, P_{E_F}]]_* \}. \quad (64)$$

The tensor is antisymmetric, so that $\sigma_{\alpha\alpha} = 0$, for $\alpha = 1, \dots, d$. If, in addition to (57), the magnetic vector potential in (53) vanishes, $A_\omega = 0$, so that the Hamiltonian H_ω is time-reversal invariant, all components of the conductivity tensor vanish.

Hence, all components of the DC conductivity vanish for time-reversal invariant systems (Hamiltonians (53) with $A = 0$) provided the Fermi projector satisfies the decay hypothesis (57). We note that for lattice models, this condition is

$$\mathbb{E} \left\{ \sum_{x \in \mathbb{Z}^d} |x|^2 |\langle 0 | P_{E_F} | x \rangle|^2 \right\} < \infty, \quad (65)$$

where $|y\rangle$ denoted the discrete delta function at $y \in \mathbb{Z}^d$.

For systems with nontrivial magnetic fields, the off-diagonal terms may be nonzero. Of special interest is the two-dimensional case with a nonzero, constant magnetic field. In this case, the integer quantum Hall effect is the fact that the off-diagonal term σ_{12} is an integer multiple of e/h when the Fermi energy E_F is in the region of localized states (see [5, 3] for lattice models and [21, 22] for models in the continuum).

4.1.2. *Relation to the ccc-measure.* The formula derived from rigorous linear response theory (62) can be manipulated in order to derive the $T = 0$ formula (5). We write (62) as

$$\sigma_{\alpha,\beta}(\eta; E_F) = -2i \int_0^\infty d\tau e^{-\eta\tau} \mathcal{T}(v_\alpha e^{-i\tau H} [x_\beta, P_{E_F}] e^{i\tau H}). \quad (66)$$

We use the spectral theorem $H = \int \lambda dE_\lambda$, and the identity

$$[x_\beta, f(H)] = i \int_{\mathbb{R}} \hat{f}(s) \int_0^s du e^{-i(s-u)H} v_\beta e^{-iuH}. \quad (67)$$

Recalling that the Fermi projector $P_{E_F} = n_F(H; 0)$, we formally write the trace-per-unit volume as

$$\mathcal{T}(v_\alpha e^{-i\tau H} [x_\beta, P_{E_F}] e^{i\tau H}) = \int \int \frac{n_F(\lambda; 0) - n_F(\nu; 0)}{\lambda - \nu} \mathcal{T}(v_\alpha dE_\nu v_\beta dE_\lambda) \quad (68)$$

Substituting (68) into (66), and performing the time integration with use of definition (6), we obtain

$$\begin{aligned} \sigma_{\alpha,\beta}(E_F) &= \lim_{\epsilon \rightarrow 0} \int \int \frac{n_F(\lambda; 0) - n_F(\nu; 0)}{\lambda - \nu} \delta_\epsilon(\lambda - \nu) \mathcal{T}(v_\alpha dE_\nu v_\beta dE_\lambda) \\ &= \lim_{\epsilon \rightarrow 0} \int \int \frac{n_F(\lambda; 0) - n_F(\nu; 0)}{\lambda - \nu} \delta_\epsilon(\lambda - \nu) M_{\alpha,\beta}(d\nu, d\lambda), \end{aligned} \quad (69)$$

where $M_{\alpha,\beta}$ is the ccc-measure.

4.2. Results on DC Conductivity. We review several papers concerning the conductivity in one-particle systems. Typically, these authors assume a formula for the conductivity and then show that under a condition such as (57) or (65) the DC conductivity vanishes.

4.2.1. *Fröhlich-Spencer and Decay of the Green's Function.* In their fundamental paper of 1983, Fröhlich and Spencer [17] proved the absence of diffusion and the vanishing of the DC conductivity for the lattice Anderson model. The authors assumed a form of the Kubo formula that expresses the DC conductivity of a gas of noninteracting electrons at $T = 0$ and Fermi energy E_F in

terms of the Green's function. Let $\rho(E) \geq 0$ be the density of states for H_ω . For $\alpha, \beta = 1, \dots, d$, the conductivity tensor is given by

$$\tilde{\sigma}_{\alpha,\beta}^{DC}(E_F)\rho(E_F) = \lim_{\eta \rightarrow 0} \frac{2\eta^2}{d\pi} \sum_{x \in \mathbb{Z}^d} x_\alpha x_\beta \mathbb{E}\{|G(0, x; E_F + i\eta)|^2\}. \quad (70)$$

Fröhlich and Spencer showed that if E_F lies in an energy interval for which the multiscale analysis (MSA) holds, then $\sigma_{\alpha,\beta}^{DC}(E_F) = 0$, assuming $\rho(E_F) > 0$ (see [26, 35]).

THEOREM 4. [17] *Suppose the Hamiltonian $H_\omega = L + V_\omega$ on $\ell^2(\mathbb{Z}^d)$, where L is the lattice Laplacian (in particular, the magnetic field is zero). Suppose that bound (71) holds for Lebesgue almost all energy in (a, b) . Then if $E_F \in (a, b)$, the DC conductivity defined in (70) at Fermi energy E_F vanishes.*

The key MSA bound on the Green's function is that, with probability one,

$$\sup_{\eta > 0} |G_\omega(x, y; E + i\eta)| \leq C_I e^{-m_I |x-y|}. \quad (71)$$

The exponential decay of the Green's function then implies that $\sigma_{\alpha,\beta}^{(DC)} = 0$. Formula (70) is equivalent to (69). The following derivation is presented, for example, in [3]:

$$\begin{aligned} & \frac{\eta^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_\alpha x_\beta \mathbb{E}\{|G(0, x; E_F + i\eta)|^2\} \\ &= -\frac{\eta^2}{\pi} \mathbb{E}\{\langle 0 | [x_\alpha, (H - E_F + i\eta)^{-1}] [x_\beta, (H - E_F - i\eta)^{-1}] | 0 \rangle\} \\ &= \frac{\eta^2}{\pi} \mathcal{T}\{R(E_F - i\eta)v_\alpha R(E_F - i\eta)R(E_F + i\eta)v_\beta R(E_F + i\eta)\} \\ &= \int \int \delta_\eta(\lambda - E_F)\delta_\eta(\nu - E_F)M_{\alpha,\beta}(d\lambda, d\nu). \end{aligned} \quad (72)$$

If, as in section 1, we assume the existence of a ccc-density $m_{\alpha,\beta}$, we may compute the limit $\eta \rightarrow 0$ in (72). We then obtain $\tilde{\sigma}_{\alpha,\beta}^{DC}(E_F) = m_{\alpha,\beta}(E_F, E_F)$. So, by (8), this gives the same conductivity as defined in (5)–(7).

4.2.2. Kunz. Kunz [29] considered the two-dimensional Landau Hamiltonian with a random potential on $L^2(\mathbb{R}^2)$ so that $H_\omega = (1/2)(i\nabla + A_0)^2 + V_\omega$, where $A_0 = B(x_2, 0)$ and the velocity operators $v_j = i[H_\omega, x_j]$. He assumed that the conductivity is given by a Kubo formula that he wrote as

$$\sigma_{1,2}(E_F) = \lim_{\epsilon \rightarrow 0} \frac{1}{i\epsilon} \int_0^\infty e^{-\epsilon t} \mathcal{T}(P_{E_F}[v_1, e^{itH_\omega} v_2 e^{-itH_\omega}] P_{E_F}) dt, \quad (73)$$

and for the diagonal terms

$$\sigma_{jj}(E_F) = \lim_{\epsilon \rightarrow 0} \frac{1}{i\epsilon} \int_0^\infty e^{-\epsilon t} \mathcal{T}(P_{E_F}[v_j, e^{itH_\omega} v_j e^{-itH_\omega}] P_{E_F}) dt + \frac{1}{\epsilon} \mathcal{T}(P_{E_F}), \quad (74)$$

for $j = 1, 2$. Note that (73) and (74) follow directly from the above Kubo formula (62), with $\xi_{E_F} = P_{E_F}$, after integrating by part (and a change of variable $t \rightarrow -t$). Assuming band-edge localization for H_ω , he shows that if the disorder is weak enough relative to B so that there is a gap between the

Landau bands and if the Fermi energy E_F lies in the gap between the n^{th} and $(n+1)^{\text{st}}$ -bands, then the transverse conductivity $\sigma_{1,2}(E_F)$ is a universal multiple of $(n+1)$. He also provides arguments for the localization length to diverge in each Landau band.

4.2.3. *Bellissard, van Elst, and Schultz-Baldes's Work on Lattice Models.* Bellissard, van Elst, and Schulz-Baldes [5] derived (64) in a one-electron model using a *relaxation time approximation*. The one-particle Hamiltonian differs from (54) in that a time-dependent perturbation $W_{\text{coll}}(t)$ is added that mimics a dissipation process. The interaction has the form $W_{\text{coll}}(t) = \sum_{k \in \mathbb{Z}} W_k \delta(t - t_k)$. The ordered collision times t_k are Poisson distributed so that $\tau_k \equiv t_k - t_{k-1}$ are independent, identically distributed random variables with an exponential distribution and mean collision time $\tau = \mathbb{E}(\tau_k)$. The amplitudes W_k are the collision operators that are assumed to commute with H and be random operators. The model is discrete. They computed the time-averaged current using this evolution and found

$$J_{\beta,\mu,\mathcal{E}}(\delta) = \sum_{i=1}^2 \mathcal{E}_i \mathcal{T}([x_i, n_F(H; T)] \frac{1}{\delta + \tilde{\tau} - \mathcal{L}_H - \mathcal{E} \cdot \nabla} [x, H]). \quad (75)$$

where $\mathcal{L}_H(A) = i[H, A]$ is the (bounded) Liouvillian. The authors then neglected the $\mathcal{E} \cdot \nabla$ term in the resolvent appearing in (75) and took the limit $\delta \rightarrow 0$. This exists provided the collision factor $\tilde{\tau} > 0$. This factor is proportional to $1/\tau$, the relaxation time. Upon differentiating with respect to the electric field, they obtained

$$\sigma_{\alpha,\beta} = \mathcal{T}([x_\alpha, n_F(H; T)] \frac{1}{\tilde{\tau} - \mathcal{L}_H} [x_\beta, H]). \quad (76)$$

In certain cases, the temperature T can be taken to zero and the relaxation time can be taken to infinity so $\tilde{\tau} \rightarrow 0$. For example, they proved that for the Landau Hamiltonian with a random potential in two-dimensions, the off-diagonal conductivity $\sigma_{1,2}^{DC}(E)$ agrees with Kubo-Středa formula (64) and is a constant on energy intervals where (65) holds. Moreover, it is an integer multiple of e^2/h .

4.2.4. *Aizenman-Graf's Work on Lattice Models.* Stimulated by the integer quantum Hall effect, Aizenman and Graf [3] considered the analog of the randomly perturbed Landau Hamiltonian on the lattice. In this case, the DC conductivity for the system with a constant magnetic field is given by the Kubo-Středa formula:

$$\sigma_{\alpha,\beta}^{DC}(E) = i \text{Tr}\{P_E[[x_\alpha, P_E], [x_\beta, P_E]]\}, \quad (77)$$

provided the operators are trace class. The exponential decay of the kernel of the Fermi projector (84) implies (65).

4.2.5. *Nakano's Result for Lattice Models.* The vanishing of the diagonal terms of the DC conductivity tensor for random, ergodic Schrödinger operators on the lattice \mathbb{Z}^d was also proven by Nakano [31] provided the Fermi projector satisfies (65).

Nakano defined two (scalar) conductivities associated with the x_1 -direction (any coordinate direction can be used) differing in the order in which certain

limits are taken. Let $H_{\omega, \mathcal{E}} = H_{\omega} + \mathcal{E}x_1$, corresponding to the Anderson Hamiltonian H_{ω} with an electric field in the x_1 -direction. The current of the system at zero temperature and at time t is given by

$$J_1(t; \mathcal{E}; E_F) = \mathcal{T}(e^{itH_{\omega, \mathcal{E}}} v_1 e^{-itH_{\omega, \mathcal{E}}} P_{E_F}), \quad (78)$$

where $v_1 = i[H_{\omega}, x_1]$. The corresponding conductivity is given by

$$\tilde{\sigma}_a(t; E_F) \equiv \lim_{\mathcal{E} \rightarrow 0} \frac{1}{\mathcal{E}} J_1(t; \mathcal{E}; E_F). \quad (79)$$

consistent with linear response theory. The first definition of the conductivity $\sigma_a(E_F)$ is the time average of the conductivity $\tilde{\sigma}_a(t; E_F)$ defined in (79):

$$\sigma_a(E_F) \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \tilde{\sigma}_1(t; E_F). \quad (80)$$

The second definition of Nakano replaces the time-average by an Abel limit:

$$\sigma_b(E_F) \equiv \lim_{\delta \rightarrow 0} \lim_{\mathcal{E} \rightarrow 0} \frac{1}{\mathcal{E}} \int_0^{\infty} dt \delta e^{-\delta t} J_1(t; \mathcal{E}; E_F). \quad (81)$$

This is the same as the result derived in [5] using the relaxation time approximation.

THEOREM 5. *Under the assumption (65) on the Fermi projector P_{E_F} , with $|x|$ replaced by $|x_1|$, we have*

$$\sigma_a(E_F) = \sigma_b(E_F) = 0. \quad (82)$$

4.3. Localization and the Decay of the Fermi Projector. When does the hypothesis (57) for continuum models, or (65) for lattice models, hold? Aizenman and Graf [3] used the method of fractional moments, developed in [4] and [1] for lattice models on $\ell^2(\mathbb{Z}^d)$, to prove the decay of the Fermi projector (65) provided the Fermi energy is in the strong localization regime.

The basic hypothesis is that the fractional moment estimate holds:

$$\sup_{\eta > 0} \mathbb{E}\{|G(x, y; E + i\eta)|^s\} \leq C_s e^{-s\mu|x-y|}, \quad x, y \in \mathbb{Z}^d, \quad (83)$$

uniformly for $\eta > 0$, for some $0 < s < 1$ and $\mu > 0$. If (83) holds at energy E_F , then they prove that the kernel of the Fermi distribution at $T = 0$ satisfies

$$\mathbb{E}\{\langle x | P_{E_F} | y \rangle\} \leq C_0 e^{-\mu|x-y|}. \quad (84)$$

As pointed out by Aizenman and Graf, this exponential decay estimate on the Fermi projector requires only that E_F lie in a regime of energies for which the fractional moment bound (83) holds. It does not require that (83) hold at all energies below E_F . The fractional moment bound (83) is known to hold in the strong localization regime or at extreme energies, cf. [1, 4].

At last, we mention that fast decay of the kernel of the Fermi projection at a given energy E turns out to be equivalent to $E \in \Xi^{CL}$ [20].

Acknowledgment. PDH thanks JMC for an invitation to CPT and FG for an invitation to Université de Cergy-Pontoise where some of this work was done. The authors thank le Centre Interfacultaire Bernoulli and acknowledge partial

support from the Swiss NSF during the time this work was completed. The authors thank L. Pastur for many enjoyable and stimulating discussions. JMC and FG were supported in part by ANR 08 BLAN 0261. PDH was supported in part by NSF grant 0803379. We thank the referees for several helpful comments.

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