

MAXIMAL ORDER OF GROWTH FOR THE RESONANCE COUNTING FUNCTIONS FOR GENERIC POTENTIALS IN EVEN DIMENSIONS

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ABSTRACT. We prove that the resonance counting functions for Schrödinger operators $H_V = -\Delta + V$ on $L^2(\mathbb{R}^d)$, for $d \geq 2$ even, with generic, compactly-supported, real- or complex-valued potentials V , have the maximal order of growth d on each sheet Λ_m , $m \in \mathbb{Z} \setminus \{0\}$, of the logarithmic Riemann surface. We obtain this result by constructing, for each $m \in \mathbb{Z} \setminus \{0\}$, a plurisubharmonic function from a scattering determinant whose zeros on the physical sheet Λ_0 determine the poles on Λ_m . We prove that the order of growth of the counting function is related to a suitable estimate on this function that we establish for generic potentials. We also show that for a potential that is the characteristic function of a ball, the resonance counting function is bounded below by $C_m r^d$ on each sheet Λ_m , $m \in \mathbb{Z} \setminus \{0\}$.

1. INTRODUCTION

We study the distribution of scattering poles or resonances of Schrödinger operators $H_V = -\Delta + V$ on $L^2(\mathbb{R}^d)$, for $d \geq 2$ and even, and real- or complex-valued, compactly-supported potentials $V \in L_0^\infty(\mathbb{R}^d; \mathbb{C})$. Let $\chi_V \in C_0^\infty(\mathbb{R}^d)$ be a smooth, compactly-supported function satisfying $\chi_V V = V$, and denote the resolvent of H_V by $R_V(\lambda) = (H_V - \lambda^2)^{-1}$ for $0 < \arg \lambda < \pi$. In the even-dimensional case, the operator-valued function $\chi_V R_V(\lambda) \chi_V$ has a meromorphic continuation to Λ , the infinitely-sheeted Riemann surface of the logarithm. We denote by Λ_m the m^{th} open sheet consisting of $z \in \Lambda$ with $m\pi < \arg z < (m+1)\pi$. We are interested in the number of poles $n_{V,m}(r)$, counted with multiplicity, of the continuation of the truncated resolvent $\chi_V R_V(\lambda) \chi_V$ on Λ_m of modulus at most $r > 0$. The *order of growth* of the resonance counting function $n_{V,m}(r)$ for H_V on the m^{th} -sheet is defined by

$$(1) \quad \rho_{V,m} \equiv \limsup_{r \rightarrow \infty} \frac{\log n_{V,m}(r)}{\log r}.$$

It is known that $\rho_{V,m} \leq d$ for $d \geq 2$ even [25, 26]. We prove that generically (in the sense of Baire typical) the resonance counting function has the maximal order of growth d on each non-physical sheet.

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Theorem 1.1. *Let $d \geq 2$ be even, and let $K \subset \mathbb{R}^d$ be a fixed, compact set with nonempty interior. There is a dense G_δ set $\mathcal{V}_F(K) \subset L_0^\infty(K; F)$, for $F = \mathbb{R}$ or $F = \mathbb{C}$, such that if $V \in \mathcal{V}_F(K)$, then $\rho_{V,m} = d$ for all $m \in \mathbb{Z} \setminus \{0\}$.*

This result is the even-dimensional analog of our previous result [5] in the odd-dimensional case. Roughly speaking, the theorem states that most Schrödinger operators H_V have the right number of resonances on each non-physical sheet. As in [2], the proof depends upon the construction of a plurisubharmonic function from which we can recover the order of growth of the resonance counting function on Λ_m . This is more difficult than in the odd-dimensional case, where one can use a Weierstrass factorization on the plane to help understand the relation between the order of growth of the determinant of the scattering matrix and the order of growth of its zero-counting function. Moreover, the even dimensional case requires the study of the poles of the resolvent far from the physical sheet. For this, one cannot simply use the determinant of the scattering matrix on the physical sheet.

In order to implement the argument in [2], we need upper bounds on $n_{V,m}(r)$ for all such V . In even dimensions, the following upper bounds for any $V \in L_0^\infty(\mathbb{R}^d; F)$ were proven by Vodev [25, 26]. Let λ_j be the poles of the continuation of $\chi_V R_V(\lambda) \chi_V$, listed with multiplicity. Vodev considered the following counting function:

$$(2) \quad N_V(r, a) \equiv \{\lambda_j \in \Lambda : 0 < |\lambda_j| \leq r, |\arg \lambda_j| \leq a\}.$$

We use the notation $\langle w \rangle \equiv (1 + |w|^2)^{1/2}$. In our setting, Vodev proves that

$$(3) \quad N_V(r, a) \leq Ca(\langle r \rangle^d + (\log a)^d), \text{ for all } r, a > 1.$$

This implies that $n_{V,m}(r) \leq C_m \langle r \rangle^d$ on each sheet $m \neq 0$.

We also need an example of a potential in our class for which the order of growth of the resonance counting function is d on each sheet Λ_m , $m \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

Theorem 1.2. *Let $V(x) = V_0 \chi_{[0,1]}(|x|)$, with $V_0 > 0$, be a real, spherically symmetric, step potential. For each $m \in \mathbb{Z} \setminus \{0\}$, there is a constant $C_m > 0$, so that for r sufficiently large,*

$$(4) \quad n_{V,m}(r) \geq C_m r^d + \mathcal{O}(r^{d-1}).$$

Asymptotics of the resonance counting function for Schrödinger operators in odd dimension with certain radial potentials were proved in [31], see also [22].

The existence of resonances for H_V in even dimensions was first proved by Tang and Sá Barreto [23]. They proved for $d \geq 4$ that there must be at least one resonance if $V \in C_0^\infty(\mathbb{R}^d; \mathbb{R})$ and V is nontrivial. This was later strengthened by Sá Barreto [18], who proved, for the same class of potentials, a global lower bound on the number of resonances. Sá Barreto defined $N_V(r)$ to be the number of poles λ of the continued, truncated resolvent, including multiplicities, satisfying $1/r < \lambda < r$, with $|\arg \lambda| < \log r$. He proved that $N_V(r)$ grows more quickly than $\log r / (\log \log r)^p$, for any $p > 1$, proving that there are infinitely-many resonances. His work does not establish the existence of infinitely-many resonances on each

non-physical sheet, a result that follows from Theorem 1.1, for “generic” potentials. In contrast, there are examples of complex-valued potentials V in dimension $d \geq 2$ for which there are no resonances of H_V away from the origin [3].

Earlier literature on the distribution of resonances for Schrödinger operators in the even-dimensional case includes Intissar’s paper [8]. Intissar defined, for any $\epsilon > 0$ and $r > 1$, a resonance counting function $N(\epsilon, r) \equiv \#\{\lambda_j \mid r^{-\epsilon} < |\lambda_j| < r^\epsilon, |\arg \lambda_j| < \epsilon \log r\}$. For even dimensions $d \geq 4$, and for any $\epsilon \in (0, \sqrt{2}/2)$, he proved the non-optimal polynomial upper bound $N(\epsilon, r) \leq C_\epsilon \langle r \rangle^{d+1}$. Zworski [32] proved a Poisson formula for resonances in even dimensions. Shenk and Thoe [19] proved the meromorphic continuation of the S -matrix for potential scattering onto Λ . In a related work, Tang [24] proved the existence of infinitely-many resonances for metric scattering in even dimensions.

For surveys of related results in odd dimensions, we refer the reader to [33] and [27], in addition to the references in [5]. We refer the reader to [4] for generic-type results for resonances in other situations. The notion of Baire typical or generic potentials was used by B. Simon in the study of singular continuous spectra for Schrödinger operators in [20].

1.1. Contents. In section 2, we first reduce the problem of counting resonances on the m^{th} -sheet to one of counting the zeros of a holomorphic function $f_m(\lambda)$ in the upper-half complex plane. We then review aspects of the theory of plurisubharmonic functions. Section 3 is devoted to the construction of a plurisubharmonic function associated with $f_m(\lambda)$ and families of potentials $V(z; x)$, parameterized by $z \in \Omega \subset \mathbb{C}^{d'}$. The main technical result on the order of growth of the counting function for the zeros of $f_m(z, \lambda)$ is Theorem 3.8. The main theorem on the distribution of resonances is proved in section 4. The proof of Theorem 1.2 on the lower bound on the number of resonances on each sheet Λ_m , $m \in \mathbb{Z} \setminus \{0\}$, for the step potential with the correct exponent d , is given in section 5. The appendix in section 6 contains the details of the uniform asymptotic expansions of Bessel and Hankel functions, due to Olver [13, 14, 15], and their application to the location of the zeros.

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2. REDUCTION TO AN OPERATOR ON Λ_0 AND OTHER PRELIMINARIES

For $0 < \arg \lambda < \pi$, we denote by $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$, where Δ is the non-positive Laplacian on \mathbb{R}^d . It is well-known that for $d \geq 4$ even and any compactly-supported, smooth function $\chi \in C_0^\infty(\mathbb{R}^d)$, the cut-off resolvent $\chi R_0(\lambda) \chi$ has an analytic continuation to Λ , the logarithmic cover of the plane (see, for example, [10, 19]). For $d = 2$ there is a logarithmic singularity at $\lambda = 0$. This is easily seen from the representation

$$(5) \quad \chi R_0(\lambda) \chi = E_1(\lambda) + (\lambda^{d-2} \log \lambda) E_2(\lambda),$$

where $E_1(\lambda)$ and $E_2(\lambda)$ are entire operator-valued functions and for $d = 2$, the operator $E_2(\lambda = 0)$ is a finite-rank operator. This representation follows from the formula for the Green's function

$$(6) \quad R_0(\lambda) = \frac{i}{4} \left(\frac{\lambda}{2\pi|x-y|} \right)^{(d-2)/2} H_{(d-2)/2}^{(1)}(\lambda|x-y|),$$

where the Hankel function of the first kind is defined by $H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z)$, and the expansion of the Neumann Bessel function $N_\nu(z)$, when $\nu \in \mathbb{N}$. We shall continue to denote this continuation by $\chi R_0(\lambda)\chi$. We shall use the following key identity, for $m \in \mathbb{Z}$, that follows from (6) and (74),

$$(7) \quad R_0(e^{im\pi}\lambda) = R_0(\lambda) - mT(\lambda)$$

where $T(\lambda)$ has Schwartz kernel

$$(8) \quad T(\lambda, x, y) = i\pi(2\pi)^{-d}\lambda^{d-2} \int_{\mathbb{S}^{d-1}} e^{i\lambda(x-y)\cdot\omega} d\omega$$

[10, Section 1.6]. We note that for any $\chi \in C_0^\infty(\mathbb{R}^d)$, $\chi T(\lambda)\chi$ is a holomorphic trace-class valued operator for $\lambda \in \mathbb{C}$.

Let $V \in L_0^\infty(\mathbb{R}^d)$, where we suppress the notation F when there is no need to distinguish real- or complex-valued potentials. When $0 < \arg \lambda < \pi$, we denote by $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$. Let $\chi_V \in C_0^\infty(\mathbb{R}^d)$ be a compactly supported function with $\chi_V V = V$. The resonances are the poles of the meromorphic continuation of $\chi_V R_V(\lambda)\chi_V$ to Λ . The set of resonances is independent of the cut-off function in the class described here. By the second resolvent formula, we have

$$(9) \quad \chi_V R_V(\lambda)\chi_V(1 + VR_0(\lambda)\chi_V) = \chi_V R_0(\lambda)\chi_V,$$

and the poles of $\chi_V R_V(\lambda)\chi_V$ correspond, with multiplicity, to the zeros of $I + VR_0(\lambda)\chi_V$.

We can reduce the analysis of the zeros of the continuation of $I + VR_0(\lambda)\chi_V$ to Λ_m to the analysis of zeros of a related operator on Λ_0 . This is very similar to the technique used by Froese in odd dimensions in [7]. Using (7), if $0 < \arg \lambda < \pi$ and $m \in \mathbb{Z}$, then $e^{im\pi}\lambda \in \Lambda_m$, and

$$\begin{aligned} I + VR_0(e^{im\pi}\lambda)\chi &= I + V(R_0(\lambda) - mT(\lambda))\chi_V \\ &= (I + VR_0(\lambda)\chi_V)(I - m(I + VR_0(\lambda)\chi_V)^{-1}VT(\lambda)\chi_V). \end{aligned}$$

For any fixed $V \in L_0^\infty(\mathbb{R}^d)$, there are only finitely many poles of $(I + VR_0(\lambda)\chi_V)^{-1}$ with $0 < \arg \lambda < \pi$. Thus

$$(10) \quad f_m(\lambda) = \det(I - m(I + VR_0(\lambda)\chi_V)^{-1}VT(\lambda)\chi_V)$$

is a holomorphic function of λ when $0 < \arg \lambda < \pi$ and $|\lambda| > c_0(\|V\|_{L^\infty})$. Moreover, with at most a finite number of exceptions, the zeros of $f_m(\lambda)$ with $0 < \arg \lambda < \pi$ correspond, with multiplicity, to the poles of $\chi_V R_V(\lambda)\chi_V$ with $m\pi < \arg \lambda < (m+1)\pi$. Henceforth, we will consider the function $f_m(\lambda)$, for $m \in \mathbb{Z} \setminus \{0\}$, in a neighborhood of Λ_0 . We write \mathbb{Z}^* for $\mathbb{Z} \setminus \{0\}$.

We mention certain symmetries of the resonances, identified as poles of the meromorphic continuation of the S -matrix, on various sheets that were discussed in [19]. For d even and V real, if $\lambda \in \Lambda_m$ is a scattering pole, then $\lambda_S = -\bar{\lambda} = \overline{e^{-i\pi}\lambda} = |\lambda|e^{i(\pi-\arg\lambda)} \in \Lambda_{-m}$ is also a scattering pole. This follows from the identity for the S -matrix: $S(\bar{\lambda})^* = 2I - S(e^{i\pi}\lambda)$. This result can also be seen from (10). If $\lambda \in \Lambda_m$ is a resonance, then $\tilde{\lambda} = e^{-im\pi}\lambda \in \Lambda_0$ satisfies $f_m(\tilde{\lambda}) = 0$. For the symmetric point $\lambda_S = -\bar{\lambda} \in \Lambda_{-m}$, we set $\tilde{\lambda}_S = e^{im\pi}\lambda_S \in \Lambda_0$. Using the identity $T(-\bar{\lambda}) = -\overline{T(\lambda)}$ for $0 < \arg \lambda < \pi$, it easily follows that for V real, $f_{-m}(\tilde{\lambda}_S) = 0$. Note that in odd dimensions and V real, the scattering poles in the lower-half complex plane are symmetric about the imaginary axis as follows from the identity $S(\bar{\lambda})^* = S(-\lambda)$.

We now turn to some preliminary analysis of the cut-off resolvent and the operator $T(\lambda)$ defined in (8). We will always work with a fixed, but arbitrary, compact subset $K \subset \mathbb{R}^d$, with nonempty interior, and potentials $V \in L_0^\infty(K; F)$, for $F = \mathbb{R}$ or $F = \mathbb{C}$. Our cut-off functions $\chi \in C_0^\infty(\mathbb{R}^d)$ satisfy $\chi = 1$ on K so $\chi V = V$ for all $V \in L_0^\infty(K; F)$. Recall the notation $\langle x \rangle = (1 + |x|^2)^{1/2}$. We begin with L^2 -estimates.

Lemma 2.1. *Suppose $\chi \in C_0^\infty(\mathbb{R}^d)$ and $|\operatorname{Im} \lambda| \leq \alpha$, for any $\alpha > 0$. Then,*

$$(11) \quad \|\chi T(\lambda)\chi\|_{L^2 \rightarrow L^2} \leq C\langle \lambda \rangle^{-1},$$

where the constant C depends on α and χ . Moreover, if $V \in L_0^\infty(K; F)$, then there is a constant c_0 such that

$$(12) \quad \|(I + VR_0(\lambda)\chi)^{-1}\|_{L^2 \rightarrow L^2} \leq 2$$

when $-\pi/2 < \arg \lambda < 3\pi/2$, $|\operatorname{Im} \lambda| \leq \alpha$, and $|\lambda| \geq c_0\langle \|V\|_{L^\infty} \rangle$. Moreover, for λ in this same set,

$$\|I - (I + VR_0(\lambda)\chi)^{-1}\|_{L^2 \rightarrow L^2} \leq C\langle \|V\|_{L^\infty} \rangle \langle \lambda \rangle^{-1}.$$

In particular, the bound (12) holds on Λ_0 when $|\lambda| > c_0\langle \|V\|_{L^\infty} \rangle$. The constants C and c_0 depend on α and χ .

In particular, (12) means that for $|\lambda| \geq c_0\langle \|V\|_{L^\infty} \rangle$ and satisfying the other conditions, $I + VR_0(\lambda)\chi$ is invertible on L^2 .

Proof. The first claim is proved in [29] and published in [1, section 2, page 255]. Using that $\|\chi R_0(\lambda)\chi\|_{L^2 \rightarrow L^2} = \mathcal{O}(\langle \lambda \rangle^{-1})$ when $0 < \arg \lambda < \pi$, we have from (7) and (11) that

$$(13) \quad \|\chi R_0(\lambda)\chi\|_{L^2 \rightarrow L^2} = \mathcal{O}(\langle \lambda \rangle^{-1})$$

for $-\pi/2 < \arg \lambda < 3\pi/2$, $|\operatorname{Im} \lambda| \leq \alpha$. Then, there is a $c_0 > 0$ so that when $|\lambda| \geq c_0\langle \|V\|_{L^\infty} \rangle$, and $-\pi/2 < \arg \lambda < 3\pi/2$, $|\operatorname{Im} \lambda| \leq \alpha$, $\|VR_0(\lambda)\chi\|_{L^2 \rightarrow L^2} \leq 1/2$. For such values of λ , $\|(I + VR_0(\lambda)\chi)^{-1}\|_{L^2 \rightarrow L^2} \leq 2$. The last claim follows from the series representation of $(I + VR_0(\lambda)\chi)^{-1}$, using (13). \square

We now turn to trace norm estimates. We denote by $\|\cdot\|_1$ the trace norm and by $\|\cdot\|_2$ the Hilbert-Schmidt norm.

Lemma 2.2. *Suppose $V \in L_0^\infty(K; F)$. Then for $-\pi/4 \leq \arg \lambda \leq 5\pi/4$, $|\operatorname{Im} \lambda| \leq 2$, $|\lambda| > c_0 \langle \|V\|_{L^\infty} \rangle$, there is a $C > 0$ so that*

$$\|VT(\lambda)\chi\|_1 \leq C \langle \|V\|_{L^\infty} \rangle \langle \lambda \rangle^{d-2}$$

and

$$\left\| \frac{d}{d\lambda} VT(\lambda)\chi \right\|_1 \leq C \langle \|V\|_{L^\infty} \rangle \langle \lambda \rangle^{d-2}.$$

Moreover, there is a $c_{0,m}$ so that if $|\lambda| > c_{0,m} \langle \|V\|_{L^\infty} \rangle$, and λ is in the same set as above,

$$(14) \quad \|(I - m(I + VR_0(\lambda)\chi)^{-1}VT(\lambda)\chi)^{-1}\|_{L^2 \rightarrow L^2} \leq 2.$$

Proof. We recall that $\|AB\|_1 \leq \|A\|_2 \|B\|_2$, and that the cut-off operator $T(\lambda)$ has a factorization as

$$\begin{aligned} VT(\lambda)\chi &= V\chi T(\lambda)\chi = i\pi(2\pi)^{-d} \lambda^{d-2} V(\mathbb{E}^X(\lambda))^t \mathbb{E}^X(-\lambda) \\ \mathbb{E}^X(\lambda) : L^2(\mathbb{R}^d) &\rightarrow L^2(\mathbb{S}^{d-1}), \quad \mathbb{E}^X(\lambda, \omega, x) = \exp(-i\lambda x \cdot \omega) \chi(x). \end{aligned}$$

Since

$$\|\mathbb{E}^X(\lambda)\|_2^2 = \int \int |\mathbb{E}^X(\lambda, \omega, x)|^2 d\omega dx \leq C e^{C|\operatorname{Im} \lambda|}$$

we obtain that

$$\|VT(\lambda)\chi\|_1 \leq C \langle \|V\|_{L^\infty} \rangle |\lambda|^{d-2} e^{C|\operatorname{Im} \lambda|}$$

with perhaps a different constant C . A similar proof gives the estimate for the derivative. Using Lemma 2.1, we can find a $c_{0,m}$ so that for $|\lambda| > c_{0,m} \langle \|V\|_{L^\infty} \rangle$,

$$\|m(I + VR_0(\lambda)\chi)^{-1}VT(\lambda)\chi\|_{L^2 \rightarrow L^2} \leq 1/2.$$

Then

$$\|(I - m(I + VR_0(\lambda)\chi)^{-1}VT(\lambda)\chi)^{-1}\|_{L^2 \rightarrow L^2} \leq 2.$$

□

We also need the following result on the function $f_m(\lambda)$ defined in (10).

Lemma 2.3. *Suppose $V \in L_0^\infty(K; F)$. Then there is a constant $c_m > 0$ so that if λ lies in the set*

$$\{\lambda \in \Lambda : |\operatorname{Im} \lambda| \leq 2, \quad |\arg \lambda| < \pi/4, \quad |\lambda| \geq c_m \langle \|V\|_{L^\infty} \rangle\}$$

then

$$|f_m(\lambda e^{i\pi})/f_m(\lambda)| \leq \exp(C \langle \|V\|_{L^\infty} \rangle \langle \lambda \rangle^{d-2})$$

and

$$|f_m(\lambda)/f_m(\lambda e^{i\pi})| \leq \exp(C \langle \|V\|_{L^\infty} \rangle \langle \lambda \rangle^{d-2})$$

for some $C > 0$. The constants c_m and C are independent of V , but they depend on the set K , χ , and m .

Proof. We shall write

$$F_V(\lambda) = (I + VR_0(\lambda)\chi)^{-1}$$

to make the equations less cumbersome. Note that

$$\begin{aligned} f_m(e^{i\pi}\lambda) &= \det(I - m(I + VR_0(e^{i\pi}\lambda)\chi)^{-1}VT(\lambda e^{i\pi})\chi) \\ &= \det(I - mF_V(e^{i\pi}\lambda)VT(e^{i\pi}\lambda)\chi) \\ &= \det(I - mF_V(\lambda)VT(\lambda)\chi) \\ &\quad \times \det(I + m(I - mF_V(\lambda)VT(\lambda)\chi)^{-1}(F_V(\lambda)VT(\lambda) - F_V(\lambda e^{i\pi})VT(\lambda e^{i\pi}))\chi) \\ &= f_m(\lambda) \det(I + m(I - mF_V(\lambda)VT(\lambda)\chi)^{-1}(F_V(\lambda)VT(\lambda) - F_V(\lambda e^{i\pi})VT(\lambda e^{i\pi}))\chi). \end{aligned}$$

First, by Lemma 2.1 (12), we have for $j = 0$ or $j = 1$, $\|F_V(e^{ij\pi}\lambda)\|_{L^2 \rightarrow L^2} \leq 2$. Second, by Lemma 2.2 (14), there is a $c_{0,m} > 0$ so that when $|\lambda| \geq c_{0,m}\langle \|V\|_{L^\infty} \rangle$, $\|(I - mF_V(\lambda)VT(\lambda)\chi)^{-1}\|_{L^2 \rightarrow L^2} \leq 2$. Third, by Lemma 2.2, for $j = 0$ or $j = 1$, $\|VT(\lambda e^{ij\pi})\chi\|_1 \leq C\|V\|_{L^\infty}\langle \lambda \rangle^{d-2}$. Finally, using $|\det(1 + A)| \leq \exp(\|A\|_1)$, we find that $|f_m(e^{i\pi}\lambda)/f_m(\lambda)| \leq \exp(C\langle \|V\|_{L^\infty} \rangle\langle \lambda \rangle^{d-2})$ for some constant $C > 0$. A similar technique gives the result for $|f_m(\lambda)/f_m(e^{i\pi}\lambda)|$. \square

The following lemma will be used in proving that a function we construct is plurisubharmonic.

Lemma 2.4. *If $|\operatorname{Im} u| \leq 1$, $|\arg u| \leq \pi/4$, $|u| \geq c_m\langle \|V\|_{L^\infty} \rangle$, then*

$$\left| \int_0^\pi \log |f_m(ue^{i\theta})| d\theta - \int_0^\pi \log |f_m(|u|e^{i\theta})| d\theta \right| \leq C\langle \|V\|_{L^\infty} \rangle^2 \langle u \rangle^{d-3}.$$

Proof. We write $u = |u|e^{i\phi}$. Then

$$\begin{aligned} & \int_0^\pi \log |f_m(ue^{i\theta})| d\theta \\ &= \int_0^\pi \log |f_m(|u|e^{i\theta+i\phi})| d\theta \\ &= \int_0^\pi \log |f_m(|u|e^{i\theta})| d\theta + \int_\pi^{\pi+\phi} \log |f_m(|u|e^{i\theta})| d\theta - \int_0^\phi \log |f_m(|u|e^{i\theta})| d\theta. \end{aligned}$$

But

$$\int_\pi^{\pi+\phi} \log |f_m(|u|e^{i\theta})| d\theta - \int_0^\phi \log |f_m(|u|e^{i\theta})| d\theta = \int_0^\phi \log \left| \frac{f_m(|u|e^{i(\theta+\pi)})}{f_m(|u|e^{i\theta})} \right| d\theta$$

Using Lemma 2.3, when $|u|$ is sufficiently large,

$$\begin{aligned} \operatorname{sign}(\phi) \int_0^\phi \log \left| \frac{f_m(|u|e^{i(\theta+\pi)})}{f_m(|u|e^{i\theta})} \right| d\theta &\leq C|\phi|\langle \|V\|_{L^\infty} \rangle(1 + |u|^{d-2}) \\ &\leq C|u|^{-1}\langle \|V\|_{L^\infty} \rangle(1 + |u|^{d-2}) \end{aligned}$$

where we use the fact that $|\phi| \leq C|u|^{-1}$ when $|u|$ is large. We may also write

$$\begin{aligned} & \text{sign}(\phi) \left(\int_{\pi}^{\pi+\phi} \log |f_m(|u|e^{i\theta})| d\theta - \int_0^{\phi} \log |f_m(|u|e^{i\theta})| d\theta \right) \\ &= -\text{sign}(\phi) \int_0^{\phi} \log \left| \frac{f_m(|u|e^{i\theta})}{f_m(|u|e^{i(\theta+\pi)})} \right| d\theta \\ &\geq -C|\phi| \langle \|V\|_{L^\infty} \rangle \langle u \rangle^{d-2} \\ &\geq -C|u|^{-1} \langle \|V\|_{L^\infty} \rangle \langle u \rangle^{d-2} \end{aligned}$$

again using Lemma 2.3. This finishes the proof of the lemma. \square

3. ORDER OF GROWTH AND PLURISUBHARMONIC FUNCTIONS

We establish the main technical results in this section. It is the analog of section 2 of [2] for the even-dimensional case. In the first subsection, we review the notion of plurisubharmonic functions, pluripolar sets, and order, and prove Lemma 3.1. In the second subsection, we construct a plurisubharmonic function associated with $f_m(\lambda)$ to which we will apply this lemma in order to estimate the order of growth of the resonance counting function. The main result is Theorem 3.8.

3.1. Review of some complex analysis. For the convenience of the reader, we recall some basic notions used in [2] that can be found in the book [9]. A domain $\Omega \subset \mathbb{C}^m$ is an open connected set.

Definition 3.1. *A real-valued function $\phi(z)$ taking values in $[-\infty, \infty)$ is plurisubharmonic in a domain $\Omega \subset \mathbb{C}^m$, and we write $\phi \in PSH(\Omega)$, if:*

- ϕ is upper semicontinuous and $\phi \not\equiv -\infty$;
- for every $z \in \Omega$, for every $w \in \mathbb{C}^m$, and every $r > 0$ such that $\{z + uw : |u| \leq r, u \in \mathbb{C}\} \subset \Omega$ we have

$$(15) \quad \phi(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(z + re^{i\theta}w) d\theta.$$

A function ϕ on $\Omega \subset \mathbb{C}^m$ is *locally plurisubharmonic* on Ω if ϕ is upper semicontinuous and $\phi \not\equiv -\infty$ and if for each $z \in \Omega$ there is a radius $b(z) > 0$ such that for all $w \in \mathbb{C}^m$ so that $|w| < b(z)$, $z + we^{i\theta} \in \Omega$ and

$$(16) \quad \phi(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(z + we^{i\theta}) d\theta.$$

We recall that every locally plurisubharmonic function on a domain Ω is in $PSH(\Omega)$ [9, Proposition I.19].

Definition 3.2. *A set $E \subset \mathbb{C}^m$ is pluripolar if for each $a \in E$ there is a neighborhood V_a containing a and a function $\phi_a \in PSH(V_a)$ such that $E \cap V_a \subset \{z \in V_a : \phi_a(z) = -\infty\}$.*

Definition 3.3. For $r > 0$, let $s(r) > 0$ be a monotone increasing function of r . If

$$\limsup_{r \rightarrow \infty} \frac{\log s(r)}{\log r} = \mu < \infty,$$

then $s(r)$ is said to be of order μ .

For a function h holomorphic in

$$(17) \quad \{\lambda \in \mathbb{C} : |\lambda| \geq R, \operatorname{Im} \lambda \geq 0\}$$

and $r > R$, define $n_{+,R}(h, r)$ to be the number of zeros of h , counted with multiplicities, in the closed upper half plane with norm between R and r , inclusive.

Lemma 3.1. Let $R > 0$, let h be holomorphic in the set in (17), and suppose, in addition, that h has only finitely many zeros on the real axis. Suppose that for some $p > 0$, and for some $\epsilon > 0$, we have

$$\int_R^r \left| \frac{h'(s)}{h(s)} \right| ds = \mathcal{O}(r^{p-\epsilon}) \text{ and } \int_{-r}^{-R} \left| \frac{h'(s)}{h(s)} \right| ds = \mathcal{O}(r^{p-\epsilon}).$$

Then $n_{+,R}(h, r)$ has order p if and only if

$$\limsup_{r \rightarrow \infty} \frac{\log \int_0^\pi \log |h(re^{i\theta})| d\theta}{\log r} = p.$$

Proof. Notice that the order of $n_{+,R_1}(h, r)$ is independent of the choice of $R_1 \geq R$, since $n_{+,R_1}(h, r) = n_{+,R_2}(h, r) + \mathcal{O}(1)$ if $R_1, R_2 \geq R$. Thus we may assume, without loss of generality, that h has no zeros with norm R and none on the semi-axes $s > R$ and $s < -R$. Then, using the principle of the argument and the Cauchy-Riemann equations, for $t > R$

$$\begin{aligned} n_{+,R}(h, t) &= \frac{1}{2\pi} \operatorname{Im} \left(\int_{-t}^{-R} \frac{h'(s)}{h(s)} ds + \int_R^t \frac{h'(s)}{h(s)} ds \right) \\ &\quad + \frac{1}{2\pi} \int_0^\pi t \frac{d}{dt} \log |h(te^{i\theta})| d\theta - \frac{1}{2\pi} \int_0^\pi R \frac{d}{dR} \log |h(Re^{i\theta})| d\theta. \end{aligned}$$

Now, just as in the proof of Jensen's equality, we divide by t and integrate from R to r to obtain

$$\begin{aligned} \int_R^r \frac{n_{+,R}(h, t)}{t} dt &= \frac{1}{2\pi} \operatorname{Im} \left(\int_R^r t^{-1} \int_{-t}^{-R} \frac{h'(s)}{h(s)} ds dt + \int_R^r t^{-1} \int_R^t \frac{h'(s)}{h(s)} ds dt \right) \\ &\quad + \frac{1}{2\pi} \int_0^\pi \log |h(re^{i\theta})| d\theta - \frac{1}{2\pi} \int_0^\pi \log |h(Re^{i\theta})| d\theta \\ (18) \quad &\quad - \frac{1}{2\pi} R \log(r/R) \int_0^\pi \frac{d}{dR} \log |h(Re^{i\theta})| d\theta. \end{aligned}$$

By our assumptions,

$$\int_R^r t^{-1} \int_{-t}^{-R} \frac{h'(s)}{h(s)} ds dt = \mathcal{O}(r^{p-\epsilon}) \text{ and } \int_R^r t^{-1} \int_R^t \frac{h'(s)}{h(s)} ds dt = \mathcal{O}(r^{p-\epsilon}).$$

Thus

$$\limsup_{r \rightarrow \infty} \frac{\log \int_R^r \frac{n_{+,R}(h,t)}{t} dt}{\log r} = p$$

if and only if

$$\limsup_{r \rightarrow \infty} \frac{\log \int_0^\pi \log |h(re^{i\theta})| d\theta}{\log r} = p.$$

But $n_{+,R}(h,t)$ and $\int_R^r \frac{n_{+,R}(h,t)}{t} dt$ have the same order, proving the lemma. \square

3.2. Construction of a Plurisubharmonic Function. Let $\Omega \subset \mathbb{C}^{d'}$ be an open connected set, let $z \in \Omega$ and $x \in \mathbb{R}^d$. We let $\mathbb{Z}^* \equiv \mathbb{Z} \setminus \{0\}$. Throughout this section we assume that $V(z) = V(z, x)$ is a function which has the following properties:

- For $z \in \Omega$, $V(z, \cdot) \in L^\infty(\mathbb{R}^d)$.
- The function $V(z, x)$ is holomorphic in $z \in \Omega$.
- There is a compact set $K_1 \subset \mathbb{R}^d$ so that for $z \in \Omega$, $V(z, x) = 0$ if $x \in \mathbb{R}^d \setminus K_1$.

We refer to these three conditions as *Assumptions (V)*. In our application in section 4, we take $d' = 1$.

For $z \in \Omega$ and fixed $\chi \in C_0^\infty(\mathbb{R}^d)$, with $\chi \equiv 1$ on K_1 , we set, in analogy with (10),

$$(19) \quad f_m(z, \lambda) = \det(I - m(I + V(z)R_0(\lambda)\chi)^{-1}V(z)T(\lambda)\chi).$$

For $m \in \mathbb{Z}^*$, and $0 < \epsilon < 1$, we define the function $g_{m,\epsilon}$ by

$$(20) \quad g_{m,\epsilon}(z, u) = \int_0^\pi \log |f_m(z, ue^{i\theta})| d\theta + \log |e^{u^{d-\epsilon}}|$$

where a branch of $u^{d-\epsilon}$ is chosen so that $u^{d-\epsilon} \in \mathbb{R}$ when $u \in \mathbb{R}_+$, and $u^{d-\epsilon}$ is holomorphic for $|\arg u| < \pi/2$. We show that $g_{m,\epsilon}$ is a plurisubharmonic function on an appropriate domain in $\Omega \times \mathbb{C}$. The additional term $\log |e^{u^{d-\epsilon}}|$ in (20) is useful in order to determine certain growth properties as shown in Lemma 3.3. We will use the notation $\Omega' \Subset \Omega$ to mean that the subset $\Omega' \subset \Omega$ is open and connected, with $\overline{\Omega'} \subset \Omega$ compact.

Lemma 3.2. *For any $\Omega' \Subset \Omega$ and the constant c_0 is as in Lemma 2.1 with $\alpha = 5$, the function $g_{m,\epsilon}(z, u)$ defined in (20) is plurisubharmonic for $(z, u) \in \Omega' \times U_{\Omega'}$ where*

$$U_{\Omega'} = \{u \in \mathbb{C} : |\operatorname{Im} u| < 2, \operatorname{Re} u > c_0 \max_{z \in \Omega'} (\|V(z)\|_{L^\infty}), |\arg u| < \pi/4\}.$$

Proof. Since $u^{d-\epsilon}$ is holomorphic on $U_{\Omega'}$ (which can be identified with an unbounded rectangle in the complex plane), then $\log |e^{u^{d-\epsilon}}|$ is plurisubharmonic in $\Omega' \times U_{\Omega'}$. Moreover, by Lemma 2.1, $f_m(z, ue^{i\theta})$, defined in (19), is holomorphic on $\Omega' \times U_{\Omega'}$ when $0 \leq \theta \leq \pi$. Thus $\log |f_m(z, ue^{i\theta})|$ is plurisubharmonic on $\Omega' \times U_{\Omega'}$. It then follows by [9, Proposition I.14] that $\int_0^\pi \log |f_m(z, ue^{i\theta})| d\theta$ is either plurisubharmonic on $\Omega' \times U_{\Omega'}$ or identically $-\infty$ there. It is easy to see that it is not

identically $-\infty$. Since the sum of two plurisubharmonic functions is plurisubharmonic, we prove the lemma. \square

In order to make notation less complicated, we define, for $\Omega' \Subset \Omega$, the constant

$$V_{M,\Omega'} = \max_{z \in \Omega'} \|V(z, \cdot)\|_{L^\infty}.$$

Lemma 3.3. *Let $g_{m,\epsilon}$ be as defined by (20), and let $\Omega' \Subset \Omega$ and $U_{\Omega'}$ be as in Lemma 3.2. Then there is an $r_m > 0$ so that for $z \in \Omega'$,*

$$\left(\max_{\substack{|\operatorname{Im} u| \leq 1, \operatorname{Re} u > 0 \\ |u|=r, u \in U_{\Omega'}}} g_{m,\epsilon}(z, u) \right) > \left(\max_{\substack{|\operatorname{Im} u| = \pm 1, \operatorname{Re} u > 0 \\ |u|=r, u \in U_{\Omega'}}} g_{m,\epsilon}(z, u) \right)$$

when $r \geq \langle V_{M,\Omega'} \rangle^{1/(1-\epsilon)} r_m$. The constant r_m depends on m , K_1 and χ , but not on V .

Proof. Let $u = re^{i\phi}$, where $|\phi| < \pi/4$ and $\operatorname{Im}(re^{i\phi}) = \pm 1$. We shall show, in fact, that $g_{m,\epsilon}(z, r) > g_{m,\epsilon}(z, re^{i\phi})$, indicating that the maximum along the arc of $|u| = r$ with $|\operatorname{Im} u| \leq 1$ occurs along the positive real axis at $u = r$. Note that

$$\begin{aligned} |e^{u^{d-\epsilon}}| \upharpoonright_{u=re^{i\phi}} &= \exp(\operatorname{Re}(r^{d-\epsilon} e^{i\phi(d-\epsilon)})) \\ &= \exp(r^{d-\epsilon} \cos((d-\epsilon)\phi)). \end{aligned}$$

Thus

$$\begin{aligned} \log |e^{u^{d-\epsilon}}| \upharpoonright_{u=r} - \log |e^{u^{d-\epsilon}}| \upharpoonright_{u=re^{i\phi}} &= r^{d-\epsilon} - r^{d-\epsilon} \left(1 - \frac{(d-\epsilon)^2 \phi^2}{2} + O(\phi^4) \right) \\ &= r^{d-\epsilon} \frac{(d-\epsilon)^2 \phi^2}{2} + O(r^{d-\epsilon} \phi^4). \end{aligned}$$

Combining this with the definition of $g_{m,\epsilon}$ and results of Lemma 2.4, we see that

$$|g_{m,\epsilon}(z, r) - g_{m,\epsilon}(z, re^{i\phi}) - r^{d-\epsilon} \frac{(d-\epsilon)^2 \phi^2}{2}| \leq C (r^{d-\epsilon} \phi^4 + \langle V_{M,\Omega'} \rangle \langle r \rangle^{d-3}).$$

Since $|\phi| \approx r^{-1}$ when r is sufficiently large, we may, by choosing r_m sufficiently large, ensure that

$$\frac{r^{d-\epsilon} (d-\epsilon)^2 \phi^2}{2} > 2(C r^{d-\epsilon} \phi^4 + \langle V_{M,\Omega'} \rangle \langle r \rangle^{d-3})$$

when $r \geq \langle V_{M,\Omega'} \rangle^{1/(1-\epsilon)} r_m$. When this holds, $g_{m,\epsilon}(z, r) - g_{m,\epsilon}(z, re^{i\phi}) > 0$, proving the lemma. \square

Lemma 3.4. *For an open set $\Omega' \Subset \Omega$, there is a $\tilde{r}_{m,\epsilon}(\Omega', V)$ such that if $r \geq \tilde{r}_{m,\epsilon}(\Omega', V)$,*

$$g_{m,\epsilon}(z, r) > \max_{\substack{|\operatorname{Im} u| \leq 1, |\arg u| \leq \pi/4 \\ \operatorname{Re} u > 0, |u|=r_m (\langle V_{M,\Omega'} \rangle^{1/(1-\epsilon)} + 1)}} g_{m,\epsilon}(z, u)$$

for $z \in \Omega'$, where r_m is defined in Lemma 3.3.

Proof. Let Ω'' be an open set such that $\Omega' \Subset \Omega'' \Subset \Omega$ and $\langle V_{M,\Omega''} \rangle^{1/(1-\epsilon)} < \langle V_{M,\Omega'} \rangle^{1/(1-\epsilon)} + 1$. Since $g_{m,\epsilon}(z, u)$ is plurisubharmonic on

$$\Omega'' \times \{u \in \mathbb{C} : |\operatorname{Im} u| \leq 1, |\arg u| \leq \pi/4, \operatorname{Re} u > 0, |u| > r_m \langle V_{M,\Omega''} \rangle^{1/(1-\epsilon)}\},$$

we have

$$(21) \quad \max_{\substack{z \in \Omega', |\operatorname{Im} u| \leq 1, |\arg u| \leq \pi/4 \\ \operatorname{Re} u > 0, |u| > r_m (\langle V_{M,\Omega'} \rangle^{1/(1-\epsilon)} + 1)}} g_{m,\epsilon}(z, u) < \infty.$$

Let $z_0 \in \overline{\Omega'}$. Then we can find $R_{z_0} > \max(1, r_m (\langle V_{M,\Omega'} \rangle^{2/(2-\epsilon)} + 1))$ such that $f_m(z_0, R_{z_0} e^{i\theta})$ has no zeros for $0 \leq \theta \leq \pi$. Additionally, we can find $\Omega'_{z_0}, \Omega''_{z_0}$, both open, such that $\Omega'_{z_0} \Subset \Omega''_{z_0} \Subset \Omega''$ and $f_m(z, R_{z_0} e^{i\theta}) \neq 0$ for $z \in \Omega''_{z_0}$, $0 \leq \theta \leq \pi$. Then $\log |f_m(z, R_{z_0} e^{i\theta})|$ and its derivatives are continuous for such z and θ , since f_m is holomorphic and nonzero.

Using (18), with f_m in place of h , and R_{z_0} in place of R , for $r > R_{z_0}$

$$(22) \quad \int_0^\pi \log |f_m(z, r e^{i\theta})| d\theta \geq -\operatorname{Im} \left(\int_{R_{z_0}}^r t^{-1} \int_{-t}^{-R_{z_0}} \frac{d}{ds} f_m(z, s) f_m(z, s)^{-1} ds dt + \int_{R_{z_0}}^r t^{-1} \int_{R_{z_0}}^t \frac{d}{ds} f_m(z, s) f_m(z, s)^{-1} ds dt \right) \\ + \int_0^\pi \log |f_m(z, R_{z_0} e^{i\theta})| d\theta \\ + R_{z_0} \log(r/R_{z_0}) \int_0^\pi \frac{d}{dR} \log |f_m(z, R e^{i\theta})| \Big|_{R=R_{z_0}} d\theta.$$

There is a C_{z_0} such that for $z \in \Omega'_{z_0}$,

$$(23) \quad \int_0^\pi \log |f_m(z, R_{z_0} e^{i\theta})| d\theta + R_{z_0} \int_0^\pi \frac{d}{dR} \log |f_m(z, R e^{i\theta})| \Big|_{R=R_{z_0}} d\theta \leq C_{z_0}$$

since we stay in a region with f_m nonzero.

Next, use that

$$\frac{d}{d\lambda} \det(I + A(\lambda)) = \operatorname{tr} \left((I + A(\lambda))^{-1} \frac{d}{d\lambda} A(\lambda) \right)$$

applied to $f_m(z, s)$. Thus

$$(24) \quad \frac{\frac{d}{ds} f_m(z, s)}{f_m(z, s)} = -m \operatorname{tr} \left((I - m(I + VR_0(\lambda)\chi)^{-1} V(z)T(\lambda)\chi)^{-1} \frac{d}{ds} ((I + VR_0(s)\chi)^{-1} VT(s)\chi) \right).$$

When $\arg s = 0$ or $\arg s = \pi$ and $\langle s \rangle > c_{0,m} \langle \|V(z)\|_{L^\infty} \rangle$,

$$\|(I - m(I + VR_0(\lambda)\chi)^{-1} V(z)T(\lambda)\chi)^{-1}\|_{L^2 \rightarrow L^2} \leq C$$

and

$$\left\| \frac{d}{ds} (I + VR_0(s)\chi)^{-1} VT(s)\chi \right\|_1 \leq C \langle \|V(z)\|_{L^\infty} \rangle \langle s \rangle^{d-2}$$

by Lemma 2.2. Using this and (24), and increasing C_{z_0} if necessary, we get that

$$(25) \quad \left| \int_{R_{z_0}}^r t^{-1} \int_{-t}^{-R_{z_0}} \frac{\frac{d}{ds} f_m(z, s)}{f_m(z, s)} ds dt \right| \leq C_{z_0} \langle \|V(z)\|_{L^\infty} \rangle \langle r \rangle^{d-1}$$

and

$$(26) \quad \left| \int_{R_{z_0}}^r t^{-1} \int_{R_{z_0}}^t \frac{\frac{d}{ds} f_m(z, s)}{f_m(z, s)} ds dt \right| \leq C_{z_0} \langle \|V(z)\|_{L^\infty} \rangle \langle r \rangle^{d-1}$$

for $z \in \Omega'_{z_0}$. Using (23), (25), and (26) in (22), with perhaps a new C_{z_0} , we find that

$$\int_0^\pi \log |f_m(z, r e^{i\theta})| d\theta \geq -C_{z_0} \langle \|V(z)\|_{L^\infty} \rangle \langle r \rangle^{d-1}.$$

Thus, for $z \in \Omega'_{z_0}$,

$$g_m(z, r) \geq r^{d-\epsilon} - C_{z_0} \langle \|V(z)\|_{L^\infty} \rangle \langle r \rangle^{d-1}.$$

By construction, $\overline{\Omega'} \subset \cup_{z_0 \in \overline{\Omega'}} \Omega'_{z_0}$ so that $\{\Omega'_{z_0}\}_{z_0 \in \overline{\Omega'}}$ forms a cover of $\overline{\Omega'}$. Since $\overline{\Omega'}$ is a compact set, there is a finite subcover. Denote the corresponding z_0 's for this finite subcover z_1, z_2, \dots, z_N . If we set $C_{\Omega'} = \max C_{z_j}$, $j = 1, \dots, N$, we have

$$(27) \quad g_m(z, r) \geq r^{d-\epsilon} - C_{\Omega'} \langle \|V(z)\|_{L^\infty} \rangle \langle r \rangle^{d-1}$$

for $z \in \Omega'$. But then there is a $\tilde{r}_m(\Omega', V)$ so that the right hand side of (27) is bigger than (21) whenever $r \geq \tilde{r}_m(\Omega', V)$. This finishes the proof. \square

Lemma 3.5. *Let $g_{m,\epsilon}(z, u)$ be as in (20) and $\Omega' \Subset \Omega$. Set*

$$\tilde{M}_{m,\epsilon,\Omega'}(z, w) = \max_{\substack{|\operatorname{Im} u| \leq 1, |\arg u| \leq \pi/4 \\ r_m(\langle V_{M,\Omega'} \rangle^{1/(1-\epsilon)} + 1) \leq |u| \leq |w|}} g_{m,\epsilon}(z, u).$$

Here r_m is as in Lemma 3.3. Then $\tilde{M}_{m,\epsilon,\Omega'}$ is plurisubharmonic on

$$\Omega' \times \{w \in \mathbb{C} : \tilde{r}_{m,\epsilon}(\Omega', V) < |w|\}$$

with $\tilde{r}_{m,\epsilon}(\Omega', V)$ as in Lemma 3.4.

The proof of this lemma resembles the proof of [2, Lemma 2.1].

Proof. It is clear that $\tilde{M}_{m,\epsilon,\Omega'} \not\equiv -\infty$. Since $g_{m,\epsilon}$ is upper semicontinuous, so is $\tilde{M}_{m,\epsilon,\Omega'}$. Since $g_{m,\epsilon}(z, u)$ is plurisubharmonic on

$$(28) \quad \{|\operatorname{Im} u| \leq 1, |\arg u| \leq \pi/4, r_m(\langle V_{M,\Omega'} \rangle^{1/(1-\epsilon)} + 1) \leq |u| \leq |w|\},$$

its maximum is attained on the boundary. So suppose $|w| > \tilde{r}_{m,\epsilon}(\Omega', V)$ and $\tilde{M}_{m,\epsilon,\Omega'}(z, w) = g_{m,\epsilon}(z, u_0)$, with u_0 on the boundary of the set in (28). By Lemma 3.3 and the choice of the set over which we take the maximum of $g_{m,\epsilon}$, $-1 < \operatorname{Im} u_0 < 1$. By Lemma 3.4, $g_{m,\epsilon}(z, |w|) > g_{m,\epsilon}(z, u)$ when $|u| = r_m(\langle V_{M,\Omega'} \rangle^{1/(1-\epsilon)} + 1)$, $|\arg u| \leq \pi/4$, $|\operatorname{Im} u| \leq 1$. Hence $|u_0| = |w|$.

For $v \in \mathbb{C}^{d'+1}$ write $v = (v', v_{d'+1}) \in \mathbb{C}^{d'} \times \mathbb{C}$. There is a $\rho(z, w)$, so that for all $\theta \in [0, 2\pi]$ and all $v \in \mathbb{C}^{d'+1}$ with $|v| < \rho(z, w)$, $z + e^{i\theta}v' \in \Omega'$ and $u_0 + v_{d'+1}e^{i\theta}$ lies in the set in (28). Then, using the fact that $g_{m,\epsilon}$ is plurisubharmonic by Lemma 3.2, we have

$$\begin{aligned} \tilde{M}_{m,\epsilon,\Omega'}(z, w) &= g_{m,\epsilon}(z, u_0) \\ &\leq (2\pi)^{-1} \int_0^{2\pi} g_{m,\epsilon}(z + e^{i\theta}v', u_0 + e^{i\theta}\frac{u_0}{w}v_{d'+1})d\theta \\ &\leq (2\pi)^{-1} \int_0^{2\pi} \tilde{M}_{m,\epsilon,\Omega'}(z + e^{i\theta}v', w + e^{i\theta}v_{d'+1})d\theta. \end{aligned}$$

This shows that $\tilde{M}_{m,\epsilon,\Omega'}$ is locally plurisubharmonic at (z, w) . Since we may do the same thing for any $z \in \Omega'$, $w \in \mathbb{C}$ with $|w| > \tilde{r}_{m,\epsilon}(\Omega', V)$, $\tilde{M}_{m,\epsilon,\Omega'}$ is plurisubharmonic. \square

Lemma 3.6. *Let $\Omega' \Subset \Omega$ and let $\tilde{M}_{m,\epsilon,\Omega'}$ be as in Lemma 3.5. The function $M_{m,\epsilon,\Omega'}(z, w)$ defined on $\Omega' \times \mathbb{C}$ by*

$$M_{m,\epsilon,\Omega'}(z, w) = \begin{cases} \max(1, \tilde{M}_{m,\epsilon,\Omega'}(z, \tilde{r}_m(\Omega', V) + 1)), & \text{if } |w| \leq \tilde{r}_m(\Omega', V) + 1 \\ \max(1, \tilde{M}_{m,\epsilon,\Omega'}(z, w)), & \text{if } |w| \geq \tilde{r}_m(\Omega', V), \end{cases}$$

is plurisubharmonic on $\Omega' \times \mathbb{C}$.

Proof. This proof follows just as the proof of [2, Lemma 2.2], if one uses in addition the fact that the supremum of two plurisubharmonic functions is plurisubharmonic ([9, Proposition I.3]). \square

Note that the dependence of $M_{m,\epsilon,\Omega'}(z, w)$ on w is only through the norm $|w|$. It follows from Lemmas 3.5 and 3.6 that the function $r \mapsto M_{m,\epsilon,\Omega'}(z, r)$ is monotone increasing. The following lemma demonstrates the relationship between the order of $n_{V(z)}(r)$ and the order of $r \mapsto M_{m,\epsilon,\Omega'}(z, r)$.

Lemma 3.7. *Let $\Omega' \Subset \Omega$ and let $\rho_{m,\epsilon,\Omega'}(z)$ be the order of $r \mapsto M_{m,\epsilon,\Omega'}(z, r)$. We then have*

$$\rho_{m,\epsilon,\Omega'}(z) = \max(d - \epsilon, \text{order of } n_{V(z),m}(r))$$

for $z \in \Omega'$, where, as above, $n_{V(z),m}(r)$ is the number of resonances of $H_{V(z)}$ on Λ_m , $m \in \mathbb{Z}^$ of norm at most $r > 0$.*

Proof. We first derive a convenient expression for the order of $n_{V(z),m}(r)$. Recall that the zeros of $f_m(z, \lambda)$ with $\lambda \in \Lambda_0$ correspond, with multiplicity, to poles of $R_{V(z)}(\lambda)$ with $\lambda \in \Lambda_m$ with at most a finite number of exceptions for fixed z . Using Lemma 3.1, (25), and (26), we see that if the order of $n_{V(z),m}(r)$ exceeds $d - 1$, then it is equal to

$$(29) \quad \limsup_{r \rightarrow \infty} \frac{\log \int_0^\pi \log |f_m(z, re^{i\theta})| d\theta}{\log r}.$$

Conversely, if the limit in (29) exceeds $d - 1$, then it is equal to the order of $n_{V(z),m}(r)$. Additionally, we note that by Lemma 2.4, if the limit in (29) exceeds $d - 1$, then it is equal to

$$(30) \quad \limsup_{r \rightarrow \infty} \frac{\log \left\{ \max_{|u|=r, |\operatorname{Im} u| \leq 1, |\arg u| \leq \pi/4} \int_0^\pi \log |f_m(z, ue^{i\theta})| d\theta \right\}}{\log r}.$$

We now relate the order of $n_{V(z),m}(r)$, given by (30) provided it is greater than $d - 1$, to the order of $M_{m,\epsilon,\Omega'}(z, u)$. When r is very large,

$$(31) \quad \log |\exp(u^{d-\epsilon})| \approx r^{d-\epsilon}, \text{ for } |u| = r, |\operatorname{Im} u| \leq 1, |\arg u| \leq \pi/4.$$

We define the constant $r_{m,V,\Omega'} \equiv r_m(\langle V_{M,\Omega'} \rangle^{1/(1-\epsilon)} + 1)$ where r_m is defined in Lemma 3.3. Using (31), this constant $r_{m,V,\Omega'}$, and the definitions of $g_{m,\epsilon}$ and $M_{m,\epsilon,\Omega'}$, we obtain

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\log M_{m,\epsilon,\Omega'}(z, r)}{\log r} &= \limsup_{r \rightarrow \infty} \frac{\log \left\{ \max_{r_{m,V,\Omega'} \leq |u| \leq r, |\operatorname{Im} u| \leq 1, |\arg u| \leq \pi/4} g_{m,\epsilon}(z, u) \right\}}{\log r} \\ &= \max(d - \epsilon, \text{order of } n_{V(z),m}(r)). \end{aligned}$$

□

We now come to the main result of this section.

Theorem 3.8. *Let $\Omega \subset \mathbb{C}^{d'}$ be an open connected set, let $m \in \mathbb{Z}$, and let $V(z, x)$ satisfy the assumptions (V). If for some $z_m \in \Omega$, the function $n_{V(z_m),m}(r)$ has order d , then there is a pluripolar set $E_m \subset \Omega$ such that $n_{V(z),m}(r)$ has order d for $z \in \Omega \setminus E_m$. Moreover, if for each $m \in \mathbb{Z}^*$, there is a z_m such that $n_{V(z_m),m}(r)$ has order d , then there is a pluripolar set E such that for every $m \in \mathbb{Z}^*$, the function $n_{V(z),m}(r)$ has order d for $z \in \Omega \setminus E$.*

The proof of this theorem uses the function $M_{m,\epsilon,\Omega'}$ which we have developed. Using this function, what remains to prove resembles the proof of [9, Corollary 1.42]; see also [2, Proposition 2.3]. We include the proof for the convenience of the reader.

Proof. Let $\Omega' \Subset \Omega$ be an open connected set, with $z_m \in \Omega'$, and let $0 < \epsilon < 1$. By [9, Proposition 1.40], since $M_{m,\epsilon,\Omega'}$ is plurisubharmonic on $\Omega' \times \mathbb{C}$, there is a sequence $\{\psi_q\}$ of negative plurisubharmonic functions on Ω' so that

$$-(\rho_{m,\epsilon,\Omega'}(z))^{-1} = \limsup_{q \rightarrow \infty} \psi_q(z) \text{ for } z \in \Omega'.$$

By results of [25, 26] and Lemma 3.7, the order $\rho_{m,\epsilon,\Omega'}(z)$ of $M_{m,\epsilon,\Omega'}$ does not exceed d . Thus

$$\limsup_{q \rightarrow \infty} \left(\psi_q(z) + \frac{1}{d} \right) \leq 0$$

and

$$\limsup_{q \rightarrow \infty} \left(\psi_q(z_m) + \frac{1}{d} \right) = 0.$$

Thus by [9, Propostion 1.39], there is a pluripolar set $E'_m \subset \Omega'$ such that $\rho_{m,\epsilon,\Omega'}(z) = d$ for $z \in \Omega' \setminus E'_m$. Thus, by Lemma 3.7, the order of $n_{V(z),m}(r)$ is d for $z \in \Omega' \setminus E'_m$.

Now let $\{\Omega_j \mid j \in \mathbb{N}\}$ be a countable collection of subsets $\Omega_j \Subset \Omega$, $\cup_{j \in \mathbb{N}} \Omega_j = \Omega$, with $z_m \in \Omega_1$. Then any $z \in \Omega$ belongs to Ω_j , for some j , so z_m and z belong to Ω_j . Applying the analysis above to Ω_j , there is a pluripolar set $E_{m,j} \subset \Omega_j$ so that the order of $n_{V(z),m}(r)$ is d for $z \in \Omega_j \setminus E_{m,j}$. We define $E_m = \cup_{j \in \mathbb{N}} E_{m,j} \subset \Omega$. Then E_m is pluripolar since it is the countable union of pluripolar sets [9, Proposition 1.37] and the order of $n_{V(z),m}(r)$ is d for $z \in \Omega \setminus E_m$.

Suppose that for each $m \in \mathbb{Z}^*$ there is a z_m with $n_{V(z_m),m}(r)$ having order d . Set $E = \cup_{\mathbb{Z}^*} E_m$, where E_m is a pluripolar set such that $n_{V(z),m}(r)$ has order d for $z \in \Omega \setminus E_m$, as guaranteed by the first part of the theorem. Then for all $m \in \mathbb{Z}$, $n_{V(z),m}(r)$ has order d for $z \in \Omega \setminus E$ and E is pluripolar [9, Proposition 1.37]. \square

4. PROOF OF THEOREM 1.1

In this section, we sketch the proof of Theorem 4. The main ingredients are Theorem 3.8, the lower bound on the number of resonances for the step potential proved in section 5, and the argument of [5]. The function $f_m(\lambda)$, defined in (10) for $m \in \mathbb{Z}^*$, is analytic in a neighborhood of $\{\lambda \in \Lambda \mid 0 \leq \arg \lambda \leq \pi, \text{ and } |\lambda| > c_0 \langle \|V\|_{L^\infty} \rangle\}$, where $c_0 > 0$ is the constant in Lemma 2.1. To indicate the dependence on a particular potential V , we will write $f_{V,m}(\lambda)$. However, we use a fixed $\chi = \chi_V$, with $\chi \equiv 1$ on K . For positive constants $N, M, q > 0$ and $j > 2Nc_0$, we define subsets of $L^\infty(K; F)$ by

$$(32) \quad \begin{aligned} A_m(N, M, q, j) \equiv & \{V \in L^\infty(K; F) : \langle \|V\|_{L^\infty} \rangle \leq N, \\ & \int_0^\pi \log |f_{V,m}(re^{i\theta})| d\theta \leq Mr^q, \\ & \text{for } 2Nc_0 \leq r \leq j\}. \end{aligned}$$

Lemma 4.1. *The set $A_m(N, M, q, j) \subset L^\infty(K; F)$ is closed.*

Proof. Let $V_k \in A_m(N, M, q, j)$, such that $V_k \rightarrow V$ in the L^∞ norm. Then clearly $\langle \|V\|_{L^\infty} \rangle \leq N$. We shall use (10) and the bound

$$(33) \quad |\det(I + A) - \det(I + B)| \leq \|A - B\|_1 e^{\|A\|_1 + \|B\|_1 + 1},$$

see, for example, [21]. With $A = (I + V_j R_0(\lambda) \chi)^{-1} V_j T(\lambda) \chi$ and $B = (I + V R_0(\lambda) \chi)^{-1} V T(\lambda) \chi$, we have

$$(34) \quad \|A - B\|_1 \leq S_1 + S_2,$$

where

$$(35) \quad S_1 = \left\| \{(I + V_j R_0(\lambda) \chi)^{-1} (V - V_j) \chi R_0(\lambda) \chi (I + V R_0(\lambda) \chi)^{-1}\} V \chi T(\lambda) \chi \right\|_1,$$

and

$$(36) \quad S_2 = \|(I + V_j R_0(\lambda)\chi)^{-1}(V_j - V)\chi T(\lambda)\chi\|_1.$$

We note that when $0 \leq \arg \lambda \leq \pi$, and $2Nc_0 \leq |\lambda| \leq j$, the term $(1 + V_j R_0(\lambda)\chi)^{-1}$ is uniformly bounded by (12). Furthermore, it converges to $(1 + V R_0(\lambda)\chi)^{-1}$ in operator norm as $V_j \rightarrow V$ and we have the bound (13). Consequently, by the trace bound in Lemma 2.2, the terms S_1 and S_2 in (35)-(36) converge to zero. Since the individual trace norms are uniformly bounded, it follows from (33) that $f_{V_k, m}(\lambda) \rightarrow f_{V, m}(\lambda)$ uniformly. Consequently,

$$\int_0^\pi \log |f_{V, m}(re^{i\theta})| d\theta \leq Mr^q,$$

for r in the specified region, so that $V \in A_m(N, M, q, j)$. \square

In the next step, we characterize those $V \in L_0^\infty(K; F)$ for which the resonance counting function exponent is strictly less than the dimension d . For $N, M, q > 0$, let

$$(37) \quad B_m(N, M, q) = \bigcap_{j \geq 2Nc_0} A_m(N, M, q, j).$$

Note that $B_m(N, M, q)$ is closed by Lemma 4.1.

Lemma 4.2. *Let $V \in L_0^\infty(K; F)$, with*

$$\limsup_{r \rightarrow \infty} \frac{\log n_{V, m}(r)}{\log r} < d.$$

Then there exist $N, M \in \mathbb{N}$, $l \in \mathbb{N}$, such that $V \in B_m(N, M, d - 1/l)$.

Proof. Applying Lemma 3.1 with $h = f_{V, m}$, we find

$$\limsup_{r \rightarrow \infty} \frac{\log \int_0^\pi \log |f_{V, m}(re^{i\theta})| d\theta}{\log r} = p < d.$$

It follows that there is a $p' \geq p$, $p' < d$, and an $M \in \mathbb{N}$ such that

$$\int_0^\pi \log |f_{V, m}(re^{i\theta})| d\theta \leq Mr^{p'}$$

when $r \geq c_0(\|V\|_{L^\infty})$. Choose $l \in \mathbb{N}$ so that $p' \leq d - 1/l$ and $N \in \mathbb{N}$ so that $N \geq \|V\|_{L^\infty}$, and then $V \in B_m(N, M, d - 1/l)$ as desired. \square

Lemma 4.3. *The set*

$$\mathcal{M}_m = \left\{ V \in L_0^\infty(K; F) : \limsup_{r \rightarrow \infty} \frac{\log n_{V, m}(r)}{\log r} = d \right\}$$

is a G_δ set, and thus

$$\mathcal{M} = \bigcap_{m \in \mathbb{Z}^*} \mathcal{M}_m$$

is as well.

Proof. By Lemma 4.2, the complement of \mathcal{M}_m is contained in

$$\bigcup_{(N,M,l) \in \mathbb{N}^3} B_m(N, M, d - 1/l),$$

which is an F_σ set since it is a countable union of closed sets. By Lemma 3.1, if $V \in \mathcal{M}_m$, then $V \notin B_m(N, M, d - 1/l)$ for any $N, M, l \in \mathbb{N}$. Thus \mathcal{M}_m is the complement of an F_σ set. \square

We can now prove our theorem.

Proof of Theorem 1.1. Since Lemma 4.3 shows that \mathcal{M}_m is a G_δ set, we need only show that \mathcal{M}_m is dense in $L_0^\infty(K; F)$. To do this, we follow the proof of [2, Corollary 1.3] with appropriate modifications. We give the proof here for the convenience of the reader. Let $V_0 \in L_0^\infty(K; F)$ and let $\epsilon > 0$. By Theorem 1.2, proved in section 5, we may choose a nonzero, real-valued, spherically symmetric $V_1 \in L_0^\infty(K; \mathbb{R})$ so that $V_1 \in \mathcal{M}_m$, for all $m \in \mathbb{Z}^*$. Then consider the function $V(z) = V(z, x) = zV_1(x) + (1 - z)V_0(x)$. This satisfies the conditions of Theorem 3.8, with $V(1) = V_1$ and $V(0) = V_0$. Thus, there exists a pluripolar set $E \subset \mathbb{C}$, so that for $z \in \mathbb{C} \setminus E$, we have

$$\limsup_{r \rightarrow \infty} \frac{\log n_{V(z), m}(r)}{\log r} = d.$$

for $z \in \mathbb{C} \setminus E$. Since $E|_{\mathbb{R}} \subset \mathbb{R}$ has Lebesgue measure 0 (e.g. [16, Section 12.2]), we may find $z_0 \in \mathbb{R}$, $z_0 \notin E$, with $|z_0| < \epsilon / (1 + \|V_0\|_{L^\infty} + \|V_1\|_{L^\infty})$. Then $V(z_0) \in \mathcal{M}_m$ for all $m \in \mathbb{Z}^*$, and $\|V(z_0) - V_0\|_{L^\infty} < \epsilon$. Moreover, if V_0 is real-valued, so is $V(z_0)$. \square

5. LOWER BOUNDS ON RESONANCES FOR CERTAIN BOUNDED, COMPACTLY-SUPPORTED POTENTIALS IN EVEN DIMENSIONS

We prove Theorem 1.2 in this section. That is, we study in even dimensions the Schrödinger operator with a potential given by a multiple of the characteristic function of the unit ball, $V(x) = V_0 \chi_{[0,1]}(|x|)$, with $V_0 > 0$. We show that $n_{V, m}(r) \geq C_{d, m} r^d + O(r^{d-1})$, with $C_{d, m} > 0$. This establishes the existence of potentials $V \in L_0^\infty(\mathbb{R}^d)$, for d even, for which the resonance counting function has the maximal order of growth d on each Riemann sheet Λ_m , defined by $m\pi < \arg \lambda < (m + 1)\pi$, $m \in \mathbb{Z}^*$. Similar calculations were performed in [31] and [22] in the odd dimensional case. In particular, Stefanov considered resonances for the transmission problem with a stepwise constant wave speed in [22, section 9]. This consists in studying the resonances of $-c(|x|)^2 \Delta$, with $c(|x|) = c \neq 1$, for $|x| \leq R$, and one otherwise. Since the perturbation enters at second-order, expansions to order ν^{-1} are sufficient. For the zero-order perturbation by V considered here, more terms in the expansions are needed.

5.1. Resonances for Schrödinger Operators on a Half-line. The resonances for the spherically symmetric Schrödinger operator $H_V = -\Delta + V$, with $V = V(|x|)$ a real-valued, bounded, spherically symmetric potential, are completely determined by the resonances for each one-dimensional Schrödinger operator H_l , $l = 0, 1, 2, \dots$, on $L^2(\mathbb{R}^+)$ obtained by symmetry reduction. We are interested in solutions to $H_\ell \psi_\ell = \lambda^2 \psi_\ell$, where

$$\begin{aligned} (H_\ell \psi_\ell)(r) &= -\psi_\ell''(r) - \frac{(d-1)}{r} \psi_\ell'(r) + \frac{l(l+d-2)}{r^2} \psi_\ell(r) + V(r) \psi_\ell(r) \\ (38) \quad &= \lambda^2 \psi_\ell(r). \end{aligned}$$

Setting $\psi_\ell(r) = r^{-\frac{d-2}{2}} u_\ell(r)$, we find the equation for u_ℓ is

$$(39) \quad \left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \lambda^2 - \frac{L_d + \left(\frac{d-2}{2}\right)^2}{r^2} - V(r) \right] u_\ell(r) = 0,$$

where $L_d = \ell(\ell + d - 2)$.

For the unperturbed case $V \equiv 0$, equation (39) is a standard Bessel equation. We follow the notation of [13, 28] and define the Hankel functions as

$$(40) \quad H_s^{(1)}(z) = [J_{-s}(z) - e^{-is\pi} J_s(z)] / [i \sin s\pi],$$

and

$$(41) \quad H_s^{(2)}(z) = [e^{is\pi} J_s(z) - J_{-s}(z)] / [i \sin s\pi],$$

where these formulas are defined in the limit when $s \in \mathbb{Z}$. Note that $J_s(z) = [H_s^{(1)}(z) + H_s^{(2)}(z)]/2$. A pair of linearly independent solutions of (39) is

$$(42) \quad u_\ell^{(1)}(r) = J_{\ell + \frac{d-2}{2}}(\lambda r), \text{ and } u_\ell^{(2)}(r) = H_{\ell + \frac{d-2}{2}}^{(1)}(\lambda r).$$

The solution $u_\ell^{(1)}(r)$ is regular at $r = 0$, whereas the solution $u_\ell^{(2)}(r)$ satisfies the outgoing condition, behaving like $\sim e^{i\lambda r}/(\lambda r)^{1/2}$, for λ with $0 < \arg \lambda < \pi$, as $r \rightarrow \infty$. In light of these formulas, we let $\nu \equiv \ell + (d-2)/2$, and note that it is an integer for d even, and half an odd integer for d odd. Returning to the free radial equation (38) with $V = 0$, we define spherical solutions as

$$(43) \quad j_\nu(\lambda r) \equiv \left(\frac{\pi}{2r}\right)^{\frac{d-2}{2}} J_{\ell + \frac{d-2}{2}}(\lambda r), \text{ and } h_\nu(\lambda r) = \left(\frac{\pi}{2r}\right)^{\frac{d-2}{2}} H_{\ell + \frac{d-2}{2}}^{(1)}(\lambda r).$$

For the Schrödinger operator H_ℓ on a half-line \mathbb{R}^+ given in (38), we define the resonances of H_ℓ as poles in the meromorphic continuation of the Green's function $G_\nu(r, r'; \lambda)$, with $\nu = \ell + (d-2)/2$. These poles are independent of r and r' as they are the zeros of the continuation of a Wronskian onto Λ_m , $m \in \mathbb{Z}^*$, or \mathbb{C}^- , for d even or d odd, respectively. In the odd-dimensional case, these poles correspond to those $\lambda \in \mathbb{C}^-$ for which there is a purely outgoing solution to $H_\ell \psi_\nu = \lambda^2 \psi_\nu$ satisfying a Dirichlet boundary condition at $r = 0$.

We now consider the potential $V(r) = V_0 \chi_{[0,1]}(r)$, with $V_0 > 0$, and construct the Green's function on the physical sheet Λ_0 . Let $\Sigma(\lambda) \equiv (\lambda^2 - V_0)^{1/2}$, where

the square root is defined so that this function has branch cuts $(-\infty, -V_0^{1/2}] \cup [V_0^{1/2}, \infty)$. Because of the simple nature of the potential $V(r)$, the equation for $0 < r < 1$ is

$$(44) \quad -\psi_\nu'' - \frac{(d-1)}{r}\psi_\nu' + \frac{\ell(\ell+d-2)}{r^2}\psi_\nu = \Sigma(\lambda)^2\psi_\nu,$$

and for $r > 1$, the solution ψ_ν satisfies the free equation

$$(45) \quad -\psi_\nu'' - \frac{(d-1)}{r}\psi_\nu' + \frac{\ell(\ell+d-2)}{r^2}\psi_\nu = \lambda^2\psi_\nu.$$

We choose two linearly independent solutions, ϕ_ν and ψ_ν of (44)–(45) so that $\phi_\nu(r=0; \lambda) = 0$ and $\psi_\nu(r; \lambda) = h_\nu(\lambda r)$ for $r > 1$. The Green's function has the form

$$(46) \quad G_\nu(r, r'; \lambda) = \frac{1}{W_\nu(\lambda)} \begin{cases} \phi_\nu(r; \lambda)\psi_\nu(r'; \lambda), & r < r' \\ \phi_\nu(r'; \lambda)\psi_\nu(r; \lambda), & r > r' \end{cases},$$

where the Wronskian $W_\nu(\lambda)$, evaluated at $r = 1$, is given by

$$(47) \quad W_\nu(\lambda) = \Sigma(\lambda)j_\nu'(\Sigma(\lambda))h_\nu^{(1)}(\lambda) - \lambda j_\nu(\Sigma(\lambda))h_\nu^{(1)'}(\lambda).$$

The function $W_\nu(\lambda)$ admits an analytic continuation in λ to \mathbb{C} , for d odd, and to Λ , for d even provided ν is also even. For the case of d even and ν odd, $\Sigma(\lambda)W(\lambda)$ has an analytic continuation to Λ . This is consistent with the fact that the Green's function extends meromorphically to Λ . For $d = 2$, the Green's function has a logarithmic singularity at $z = 0$.

We conclude that a value $\lambda_0 \in \Lambda_m$, $m \neq 0$, for d even (or $\lambda_0 \in \mathbb{C}^-$ if d is odd) is a resonance if it satisfies the following condition:

$$(48) \quad \Sigma(\lambda_0)j_\nu'(\Sigma(\lambda_0))h_\nu^{(1)}(\lambda_0) = \lambda_0 j_\nu(\Sigma(\lambda_0))h_\nu^{(1)'}(\lambda_0).$$

Using the definitions (43), we rewrite this relation as

$$(49) \quad \Sigma(\lambda_0)J_\nu'(\Sigma(\lambda_0))H_\nu^{(1)}(\lambda_0) - \lambda_0 J_\nu(\Sigma(\lambda_0))H_\nu^{(1)'}(\lambda_0) = 0, \quad \nu = \ell + (d-2)/2.$$

In order to study the defining equation (49) on Λ_m , we define a function $F_m^{(\nu)}(\lambda)$ on Λ_0 by

$$(50) \quad F_m^{(\nu)}(\lambda) = \Sigma(\lambda)J_\nu'(\Sigma(\lambda))H_\nu^{(1)}(e^{im\pi}\lambda) - e^{im\pi}\lambda J_\nu(\Sigma(\lambda))H_\nu^{(1)'}(e^{im\pi}\lambda),$$

using the fact that $\Sigma(e^{im\pi}\lambda) = \Sigma(\lambda)$, for $m \in \mathbb{Z}$. It follows from the fundamental equation (49) that the zeros of $F_m^{(\nu)}(\lambda)$ on Λ_0 correspond to the resonances of the one-dimensional Schrödinger operator H_ℓ on the sheet Λ_m , for $|m| \geq 1$. We will study the zeros of $F_m^{(\nu)}(\lambda)$ on Λ_0 in sections 5.3 and 6.4 below.

We make some important observations concerning $F_m^{(\nu)}(\lambda)$. First, we note that when $V_0 = 0$, $F_m^{(\nu)}(\lambda)$ is $e^{im\pi}\lambda$ times the Wronskian of J_ν and $H_\nu^{(1)}$ evaluated at $e^{im\pi}\lambda$. This is easily seen to be equal to the constant $2i\pi$, which is consistent with the fact that there are no resonances in the free case. Secondly, when $m = 0$, corresponding to the physical sheet, there are no zeros of $F_0^{(\nu)}(\lambda)$ on Λ_0 , see (60). Thirdly, this equation reflects the symmetry properties of the meromorphic

continuation of the resolvent and S -matrix depending on whether d is odd or even. As mentioned in section 2, for $d \geq 2$ even, and V real, if $k \in \Lambda_m$, $m \in \mathbb{Z}^*$ is a zero, then so is $-\bar{k} \in \Lambda_{-m}$. In the odd-dimensional case, again with V real, if $k \in \mathbb{C}^-$ is a zero, then so is $-\bar{k} \in \mathbb{C}^-$, so the resonances are symmetric about the imaginary axis.

Remark. For $d = 3$, the poles of the S -matrix associated with the Schrödinger operator (38) with a spherically symmetric step potential barrier $V_0 > 0$ (the case considered here) or well $V_0 < 0$, were studied by Nussenzveig [12]. The S -matrix is explicitly computable. Let $\Sigma(\lambda) \equiv (\lambda^2 \pm V_0)^{1/2}$, where the plus sign is for the potential well, and the minus sign is for the barrier. For each angular momentum $\ell \geq 0$, the corresponding component of the S -matrix is

$$(51) \quad S_\ell(\lambda) = -\frac{\lambda j_\ell(\Sigma(\lambda))h_\ell^{(2)'}(\lambda) - \Sigma(\lambda)j_\ell'(\Sigma(\lambda))h_\ell^{(2)}(\lambda)}{\lambda j_\ell(\Sigma(\lambda))h_\ell^{(1)'}(\lambda) - \Sigma(\lambda)j_\ell'(\Sigma(\lambda))h_\ell^{(1)}(\lambda)}.$$

This should be compared with (49). As this formula also holds in the even-dimensional case, we see that the resonances defined via the continuation of the resolvent are the same as those defined by the continuation of the S -matrix. Nussenzveig studied the behavior of bound states $V_0 < 0$ and resonances $V_0 > 0$, describing various asymptotics expansions of their real and imaginary parts, especially in the case of $\ell = 0$.

5.2. Analysis of Zeros: an Overview. We discuss here the strategy for estimating the zeros of $F_m^{(\nu)}(\lambda)$ on the sheet Λ_0 . We prove that inside a semicircle of radius $r > 0$, the angular momentum states with $\ell < r$ contribute at least $c_m r^d$ zeros, where $c_m > 0$ depends on $m \in \mathbb{Z}^*$. We need to control the remainder term uniformly in r and ℓ as $r \rightarrow \infty$. Hence, we need to study the zeros of $F_m^{(\nu)}(\lambda)$ as $|\lambda|$ and ν go to infinity. We define new variables $z \equiv \lambda/\nu$ and $\tilde{z}(z) \equiv (z^2 - \nu^{-2}V_0)^{1/2}$, so that we study

$$(52) \quad F_m^{(\nu)}(\nu z) = \nu \tilde{z}(z) J_\nu'(\nu \tilde{z}) H_\nu^{(1)}(e^{im\pi} \nu z) - e^{im\pi} \nu z J_\nu(\nu \tilde{z}(z)) H_\nu^{(1)'}(e^{im\pi} \nu z).$$

for z in a fixed region. For this, we use the uniform large-order asymptotics of the Bessel and Hankel functions due to Olver [14, 15]. A special role in these uniform asymptotics is played by the compact, eye-shaped region K in the complex plane defined as follows. This region is bounded by the curves containing the points labeled B, P, E, E', P' in Figure 1 and [13, chapter 11]. Let t_0 be the positive root of $t = \coth t$, so $t_0 \sim 1.19967864 \dots$. The region K is the symmetric region in the neighborhood of the origin bounded in \mathbb{C}^+ by the curve

$$(53) \quad z = \pm(t \coth t - t^2)^{1/2} + i(t^2 - t \tanh t)^{1/2}, \quad 0 \leq t \leq t_0,$$

intercepting the real axis at ± 1 and intercepting the imaginary axis at iz_0 , where $z_0 = (t_0^2 - 1)^{1/2} \sim 0.66274 \dots$. The region K is bounded by the conjugate curve in the lower half-plane. The significance of this region K is illustrated by the fact

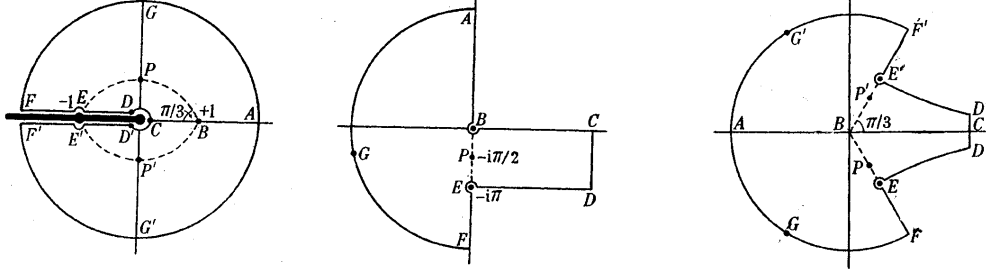


FIGURE 1. Left: the z -plane with the region K enclosed by the curves joining B, P, E, E', P' . Middle: The image of the z plane under the mapping ρ defined in (54). Right: The image of the z -plane under the mapping ζ defined in (54). Reproduced, with permission of the author and publisher, from [15, page 336].

that the ordinary Bessel function $J_\nu(z)$ decays exponentially in ν for $z \in \text{Int}(K)$, whereas it grows exponentially in ν and z for $z \in \text{Ext}(K)$.

Essential to understanding the uniform asymptotics of the Bessel and Hankel functions is the mapping on the complex plane $z \rightarrow \zeta(z)$ defined by

$$(54) \quad \rho(z) \equiv \frac{2}{3}\zeta^{3/2}(z) = \log \frac{1 + \sqrt{1 - z^2}}{z} - \sqrt{1 - z^2}.$$

The relationship between the variables z , $\zeta(z)$, and $\rho(z)$ is clarified by Figure 1 and is described in detail beginning on page 335 of Olver's article [15] (reproduced on page 420 of his book [13] where the author changed notation and used ξ for what is called ρ here and in [15]). We present these figures, with permission of the author and publisher, in Figure 1.

The cut along the negative real axis in the z -plane is mapped onto two cuts along the rays $\arg \zeta = \pm\pi/3$. The eye-shaped region K is mapped into the sector $|\arg \zeta| < \pi/3$. A neighborhood of the positive real axis in the z -plane is mapped onto a neighborhood of the negative real axis in the ζ -plane. The upper boundary of the eye-shaped region K in the z -plane is mapped onto the line segment $(0, -i\pi)$ in the ρ -plane, with the point z_0i being mapped to $-i\pi/2$. We analyze the function $F_m^{(\nu)}(\nu z)$ for large ν with z in a neighborhood of K .

5.3. Resonances on the m^{th} Sheet Λ_m , $m \in \mathbb{Z}^*$. We consider the general case $m \neq 0$ and prove a lower bound on the number of resonances. The zeros of $F_m^{(\nu)}(\nu z)$, defined in (52) for $z \in \Lambda_0$, correspond to the resonances of H_ℓ on the sheet Λ_m . We recall from section 5.2 that in order to use the uniform asymptotics of section 6.4, we defined $z \equiv \lambda/\nu$ and $\tilde{z}(z) \equiv (z^2 - \nu^{-2}V_0)^{1/2}$. We also recall that $d \geq 2$ is even so that $\nu = \ell + (d - 2)/2$ is a nonnegative integer. Using formulas

(74) and (75), we find that

$$(55) \quad F_m^{(\nu)}(\nu z) = (-1)^{m\nu} [F_0^{(\nu)}(\nu z) - 2mG_0^{(\nu)}(\nu z)],$$

where, from (52),

$$(56) \quad F_0^{(\nu)}(\nu z) = \nu \tilde{z} J'_\nu(\nu \tilde{z}) H_\nu^{(1)}(\nu z) - \nu z J_\nu(\nu \tilde{z}) H_\nu^{(1)'}(\nu z),$$

and we define

$$(57) \quad G_0^{(\nu)}(\nu z) \equiv \nu \tilde{z} J'_\nu(\nu \tilde{z}) J_\nu(\nu z) - \nu z J_\nu(\nu \tilde{z}) J'_\nu(\nu z).$$

It follows that in order to study the solutions of $F_m^{(\nu)}(\nu z) = 0$, we need to consider those z with $0 < \arg z < \pi$ for which

$$(58) \quad F_0^{(\nu)}(\nu z) = 2mG_0^{(\nu)}(\nu z).$$

It is sufficient for the lower bound to prove that for any $\nu < r$, for $r \gg 0$ sufficiently large, that there are at least $\nu(1 - \epsilon_1)$, $\epsilon_1 > 0$, solutions of this equation in the half-disk $\text{Im } \lambda > 0$ and $|\lambda| \leq r$, uniformly in r and ν .

To prove that there are at least $\nu(1 - \epsilon_1)$ zeros of $F_m^{(\nu)}(\lambda)$ near the upper boundary of the eye-shaped region νK , we concentrate a small region $\Omega_{1,\epsilon}$ near the upper boundary of K . In particular, we define, for fixed $\epsilon > 0$,

$$(59) \quad \Omega_{1,\epsilon} = \{z \in \mathbb{C}^+ : \text{dist}(z, \partial K^+) < \epsilon\} \cap \{z \in \mathbb{C}^+ : |z+1| > \epsilon \text{ and } |z-1| > \epsilon\}.$$

We recall that $\rho(z)$ is defined in (54) and that the region $\Omega_{1,\epsilon}$ near the upper boundary of K is mapped onto a neighborhood of the line segment $(-i\pi + ih(\epsilon), -ih(\epsilon))$, where $h(\epsilon) > 0$, $h(\epsilon) = \mathcal{O}(\epsilon)$, in the ρ -plane (see the middle figure of Figure 1 or Fig. 3–6 in [14]). Consequently, we will identify the zeros near the upper edge of K in the z -plane with their image $\rho(z)$ near this line segment in the ρ -plane.

We compute the uniform asymptotics of $F_0^{(\nu)}(\nu z)$ in section 6.6. From (103), we have

$$(60) \quad F_0^{(\nu)}(\nu z) = \frac{-2i}{\pi} \left\{ 1 - \frac{1}{\nu} \left[\frac{V_0(1-z^2)^{1/2}}{2z^2} \right] + \mathcal{O}\left(\frac{1}{\nu^2}\right) \right\}.$$

The uniform asymptotics of the term $G_0^{(\nu)}$ on the right of (58) is also computed in section 6.6:

$$(61) \quad G_0^{(\nu)}(\nu z) = \frac{e^{-2\nu\rho}}{2\pi} \left[\frac{V_0}{2\nu^2(1-z^2)} + \mathcal{O}\left(\frac{1}{\nu^3}\right) \right],$$

where the error is uniform for $z \in \Omega_{1,\epsilon}$. Consequently, the condition for zeros on the m^{th} -first sheet is

$$(62) \quad e^{2\nu\rho(z)} (1 + g_1(z, \nu)) = \frac{imV_0}{4\nu^2} \left(\frac{1}{1-z^2} \right) + g_2(z, \nu),$$

where $g_1(z, \nu) = \mathcal{O}(1/\nu)$, and $g_2(z, \nu) = \mathcal{O}(1/\nu^3)$, both uniformly for $z \in \Omega_{1,\epsilon}$. We note that for $V_0 = 0$ there are no solutions to this equation.

We now study the solutions of (62). The variable ρ lies in a set that is the image of $\Omega_{1,\epsilon}$ under the mapping $z \rightarrow \rho$ given in (54). This set contains a neighborhood of an interval of the negative imaginary axis of the form $(-\pi+h(\epsilon), -h(\epsilon))i \subset (-\pi, 0)i$. We will prove that there exists at least $\nu(1-\epsilon_1)$ solutions in a neighborhood of this set. We rewrite (62) as

$$(63) \quad \nu^2 e^{2\nu\rho}(1-z^2)(1+g_1(z,\nu)) = \frac{imV_0}{4} + g_3(\rho,\nu),$$

with $g_3(z,\nu) = \mathcal{O}(\nu^{-1})$, uniformly for ρ in the image of $\Omega_{1,\epsilon}$. We define two functions:

$$(64) \quad g(z,\nu) = \nu^2 e^{2\nu\rho} - \frac{imV_0}{4},$$

and

$$(65) \quad f(z,\nu) = \nu^2(1-z^2)e^{2\nu\rho}(1+g_1(z,\nu)) - \frac{imV_0}{4} - g_3(z,\nu).$$

We will prove that near each zero of $g(z,\nu)$, in a neighborhood of $(-\pi+h(\epsilon), -h(\epsilon))i \subset (-\pi, 0)i$, there is a zero of $f(z,\nu)$ using Rouché's Theorem.

Lemma 5.1. *The function $g(z,\nu)$, defined in (64) for $m \in \mathbb{Z}^*$, has infinitely-many zeros of the form*

$$(66) \quad \rho_k = \left\{ \frac{1}{2\nu} \log \left(\frac{|m|V_0}{4} \right) - \frac{\log \nu}{\nu} \right\} + i \frac{\pi}{\nu} \left[k + \operatorname{sgn}(m) \frac{1}{4} \right], \quad k \in \mathbb{Z}.$$

Proof. Viewing the function $g(z,\nu)$ as a function of ρ , it is clear that it is periodic in $\operatorname{Im} \rho$ with period π/ν . We can then explicitly solve $g(z,\nu) = 0$. \square

The values ρ_k provide the approximate solutions of $f(z,\rho) = 0$ as the next lemma shows.

Lemma 5.2. *For $m \in \mathbb{Z}^*$, in a neighborhood of each zero ρ_k of $g(z,\nu)$ with imaginary part in $(-i\pi + i2h(\epsilon), -i2h(\epsilon))$, there is exactly one zero of $f(z,\nu)$. Consequently, there are at least $\nu(1-\epsilon_1)$, $\epsilon_1 = \mathcal{O}(\epsilon) > 0$ zeros of $f(z,\nu)$ in a neighborhood of the interval on the negative imaginary axis $(-\pi, 0)i$ for all $\nu > 0$ large.*

We remark that $\epsilon_1 > 0$ may be made arbitrarily small by choosing $\epsilon > 0$ sufficiently small.

Proof. 1. We use Rouché's Theorem applied on small rectangular contour \mathcal{C} about ρ_k with vertical sides A and B , and horizontal sides C and D , chosen so that exactly one zero ρ_k of $g(z,\nu)$ lies in the rectangle. This is possible as the approximate solutions (66) are separated as $|\operatorname{Im} \rho_k - \operatorname{Im} \rho_{k+1}| = \pi/\nu$. In order to apply Rouché's Theorem, we must show that on each segment of the contour \mathcal{C} we have

$$(67) \quad |f(z,\nu) - g(z,\nu)| < |g(z,\nu)|,$$

for all ν large. We will repeatedly use the fact that for $z \in \Omega_{1,\epsilon}$, we have $|z|^2 < (1 - \delta(\epsilon))$, for some small $\delta > 0$, a function of $\epsilon > 0$ used to define $\Omega_{1,\epsilon}$ specified in (59).

2. Vertical sides A and B . Let side A lie on the negative imaginary axis and be parameterized by it , for $t < 0$. Along this side, $|g(z, \nu)| = \nu^2 + \mathcal{O}(1)$ and $|(f(z, \nu) - g(z, \nu)) \upharpoonright_A| \leq (1 - \delta)\nu^2 + \mathcal{O}(\nu^{-1})$, so we have (67). Along edge B , we let $\rho = -a + it$, for $a = \alpha \log \nu / \nu > 0$, with $\alpha > 1$, and $t < 0$. We easily find that $|(f(z, \nu) - g(z, \nu)) \upharpoonright_B| \leq (1 - \delta)\nu^{-2(\alpha-1)} + \mathcal{O}(\nu^{-1})$. On the other hand, we have $|g(z, \nu) \upharpoonright_B| = (|m|V_0/4)(1 + \mathcal{O}(\nu^{-2(\alpha-1)}))$. It follows that (67) holds along A and B .

3. Horizontal sides C and D . For zero ρ_k , with $k < 0$, we choose the straight line segment C so that on C we have

$$(68) \quad \operatorname{Im} \rho_{k-1} = \frac{\pi}{\nu} \left[k - 1 + \operatorname{sgn}(m) \frac{1}{4} \right] < \operatorname{Im} \rho < \operatorname{Im} \rho_k = \frac{\pi}{\nu} \left[k + \operatorname{sgn}(m) \frac{1}{4} \right],$$

and such that

$$(69) \quad 2\nu \operatorname{Im} \rho = \frac{\pi}{2} \pmod{2\pi}.$$

With $\operatorname{Re} \rho = t$, we then have

$$|(f(z, \nu) - g(z, \nu)) \upharpoonright_C| \leq (1 - \delta)\nu^2 e^{2\nu t} (1 + \mathcal{O}(\nu^{-1})) + \mathcal{O}(\nu^{-1}),$$

whereas $|g(z, \nu)|_C = [(\nu^2 e^{2\nu t})^2 + (|m|V_0/4)^2]^{1/2}$, so we again have (67). A similar choice of D insures the same inequality. \square

We now prove Theorem 1.2 for the case $m \in \mathbb{Z}^*$.

Proof. Recall that $\nu = l + (d-2)/2$, $l \in \mathbb{N}$ and $\epsilon_1 > 0$ is arbitrary. From Lemma 5.2 it follows that there are, for each $\nu > 0$ large, at least $\nu(1 - \epsilon_1)$ zeros near the upper edge of the eye-shaped region K . Furthermore, each zero of $F_m^{(\nu)}(\lambda)$ corresponds to a resonance of multiplicity $m(l)$, the dimension of the space of spherical harmonics on \mathbb{S}^{d-1} with eigenvalue $l(l + d - 2)$. Since $m(l) \geq cl^{d-2} + \mathcal{O}(l^{d-3})$, for some $c > 0$, it follows from Lemma 5.2 that

$$(70) \quad n_{1,V}(r) \geq \sum_{\ell=1}^{[r]} \frac{1}{4} (l - (d-2)/2) (cl^{d-2} + \mathcal{O}(l^{d-3})) \geq Cr^d + \mathcal{O}(r^{d-1}),$$

for some $C > 0$, depending on $m \in \mathbb{Z}^*$. \square

This proves the lower bound for the even-dimensional case on the m^{th} -sheet, $m \in \mathbb{Z}^*$. We recall that for V real, the symmetry of the zeros means that the resonances on Λ_{-m} are the same as those on Λ_m .

Remark. We make some historic comments relevant to the odd dimensional case. For $d = 3$, resonances of spherically-symmetric, compactly-supported potentials were studied by many physicists and Newton provided a nice summary [11]. In particular, R. Newton studied the zeros of the Jost function $f_\ell(\lambda)$ for each angular momentum component in dimension three. These are the same as the zeros of

the function $F_1^{(\nu)}(\lambda)$. For a real potential with compact support inside the ball of radius $R > 0$, he gives a proof that $f_\ell(\lambda)$ has infinitely many complex roots in the lower-half complex plane, and that these roots are symmetric with respect to the imaginary axis. This follows from the fact that $f_\ell(\lambda)f_\ell(-\lambda)$ is an entire function of λ^2 of order $1/2$. It also follows that only finitely-many roots lie on the negative imaginary axis. Finally, he proves that if λ_n is a sequence of roots with positive real parts, then $\operatorname{Re} \lambda_n = n\pi/R + \mathcal{O}(1)$ and $\operatorname{Im} \lambda_n = [(\sigma + 2)/(2R)] \log n + \mathcal{O}(1)$. In particular, this shows that there are infinitely many roots in the region $2\pi - \epsilon < \arg \lambda < 0$. These lie outside of the region considered above.

6. APPENDIX: ANALYTIC CONTINUATION AND UNIFORM ASYMPTOTICS OF BESSEL AND HANKEL FUNCTIONS

In this appendix, we provide all the details necessary for obtaining a lower bound on the number of zeros of the function $F_m^{(\nu)}(\lambda)$ on the m^{th} -sheet. In the first section we give the analytic continuation of the Bessel and Hankel functions following Olver [13, chapter 7]. We then summarize the uniform large-order asymptotics of the Bessel and Hankel functions proved by Olver [14, 15]. These rely on the asymptotic expansion of the Airy functions (see, for example, [15, appendix]) that we present in the next section. Finally, we compute the uniform asymptotics of the terms occurring in $F_m^{(\nu)}(\lambda)$.

6.1. Analytic Continuation of Bessel and Hankel Functions. The analytic continuations of the ordinary Bessel functions $J_\nu(z)$, for $\nu \in \mathbb{R}$, from the region Λ_0 to the region Λ_m , are obtained by the formula

$$(71) \quad J_\nu(ze^{im\pi}) = e^{im\nu\pi} J_\nu(z).$$

It follows that

$$(72) \quad J'_\nu(ze^{im\pi}) = e^{im\pi(\nu+1)} J'_\nu(z).$$

As for the Hankel function $H_\nu^{(1)}(z)$, and $z \in \Lambda_0$, the analytic continuation to Λ_m , with $m \in \mathbb{Z}^*$, is obtained through the following formula (e.g. [13, chapter 7]),

$$(73) \quad H_\nu^{(1)}(e^{im\pi}z) = -\frac{\sin(m-1)\nu\pi}{\sin\nu\pi} H_\nu^{(1)}(z) - e^{-i\pi\nu} \frac{\sin\nu m\pi}{\sin\nu\pi} H_\nu^{(2)}(z),$$

where, if $\nu \in \mathbb{Z}$, we define the right side by the limit.

In our case, $\nu = \ell + (d-2)/2$, with $\ell = 0, 1, 2, \dots$, so that when d is even, ν is a non-negative integer $\nu = 0, 1, 2, \dots$, and when d is odd, ν is half an odd integer. In the case when $d \geq 4$ is even, the Hankel functions are analytic on the Riemann surface of the logarithm Λ . When $d = 2$ there is a logarithmic singularity at the origin $z = 0$. In the case when $\nu \in \mathbb{Z}$, formula (73) becomes

$$(74) \quad \begin{aligned} H_\nu^{(1)}(e^{im\pi}z) &= (-1)^{m\nu+1} [(m-1)H_\nu^{(1)}(z) + mH_\nu^{(2)}(z)] \\ &= (-1)^{m\nu} [H_\nu^{(1)}(z) - 2mJ_\nu(z)]. \end{aligned}$$

As for the derivatives of the Hankel function $H_\nu^{(1)}(z)$, for $0 < \arg z < \pi$, the analytic continuation to the sheet Λ_m with $m\pi < \arg z < (m+1)\pi$, for $m \in \mathbb{Z}$, is obtained from (73). Restricting ourselves to the case of interest $\nu \in \mathbb{Z}$, we obtain:

$$\begin{aligned} H_\nu^{(1)'}(e^{im\pi}z) &= (-1)^{m(\nu+1)+1}[(m-1)H_\nu^{(1)'}(z) + mH_\nu^{(2)'}(z)] \\ (75) \qquad \qquad &= (-1)^{m\nu+1}[H_\nu^{(1)'}(z) - 2mJ_\nu'(z)]. \end{aligned}$$

6.2. Asymptotic Expansions of Bessel and Hankel Functions. The asymptotics of the Bessel and Hankel functions used here are expressed in terms of Airy functions. As in [13], we adopt the convention that $Ai(w)$ has its zeros on the negative real axis. The index ν is real and positive. It is convenient to define the following functions:

$$(76) \qquad \phi(\zeta) \equiv \left(\frac{4\zeta}{1-z^2} \right)^{1/4} = \left(-\frac{2}{z} \frac{dz}{d\zeta} \right)^{1/2},$$

and

$$(77) \qquad \chi(\zeta) \equiv \frac{\phi'(\zeta)}{\phi(\zeta)} = \frac{4 - z^2[\phi(\zeta)]^6}{16\zeta},$$

and

$$(78) \qquad \psi(\zeta) \equiv \frac{2}{z\phi(\zeta)}.$$

We also need the following series expansions:

$$(79) \qquad F_1(\zeta, \nu) = \sum_{j=0}^{\infty} \frac{A_j(\zeta)}{\nu^{2j}}, \quad F_2(\zeta, \nu) = \sum_{j=0}^{\infty} \frac{B_j(\zeta)}{\nu^{2j}},$$

with $A_0(\zeta) = 1$, and the remaining coefficients are determined recursively, see [14, section 9]. The functions $G_j(\zeta, \nu)$, $j = 1, 2$, are given as infinite series

$$(80) \qquad G_1(\zeta, \nu) = \sum_{j=0}^{\infty} \frac{C_j(\zeta)}{\nu^{2j}}, \quad G_2(\zeta, \nu) = \sum_{j=0}^{\infty} \frac{D_j(\zeta)}{\nu^{2j}},$$

where the coefficients are

$$(81) \qquad C_j(\zeta) = \chi(\zeta)A_j(\zeta) + A_j'(\zeta) + \zeta B_j(\zeta),$$

$$(82) \qquad D_j(\zeta) = \chi(\zeta)B_{j-1}(\zeta) + B_{j-1}'(\zeta) + A_j(\zeta),$$

with $B_{-1}(\zeta) = 0$ and $D_0(\zeta) = 1$.

For the ordinary Bessel functions with $|\arg z| < \pi - \epsilon$, for any $\epsilon > 0$, Olver proved that

$$(83) \qquad J_\nu(\nu z) \sim \phi(\zeta) \left(\frac{Ai(\nu^{2/3}\zeta)}{\nu^{1/3}} F_1(\zeta, \nu) + \frac{Ai'(\nu^{2/3}\zeta)}{\nu^{5/3}} F_2(\zeta, \nu) \right).$$

He also proved that the asymptotic expansion for the derivatives can be obtained by differentiation. It is useful to recall that the Airy function $Ai(w)$ (and $Ai_{\pm 1}(w)$)

introduced below) satisfies the differential equation $Ai''(u) = uAi(u)$. For the ordinary Bessel functions, one obtains for $|\arg z| < \pi - \epsilon$, and for any $\epsilon > 0$,

$$(84) \quad J'_\nu(\nu z) \sim -\psi(\zeta) \left(\frac{Ai(\nu^{2/3}\zeta)}{\nu^{4/3}} G_1(\zeta, \nu) + \frac{Ai'(\nu^{2/3}\zeta)}{\nu^{2/3}} G_2(\zeta, \nu) \right).$$

We also need the uniform asymptotics of the Hankel function. As in Olver [13], we define $Ai_{\pm 1}(w) = Ai(e^{\mp 2\pi i/3} w)$. For $|\arg z| \leq \pi - \epsilon$, for any $\epsilon > 0$, we have

$$(85) \quad H_\nu^{(1)}(\nu z) \sim 2e^{-i\pi/3} \phi(\zeta) \left(\frac{Ai_{-1}(\nu^{2/3}\zeta)}{\nu^{1/3}} F_1(\zeta, \nu) + \frac{Ai'_{-1}(\nu^{2/3}\zeta)}{\nu^{5/3}} F_2(\zeta, \nu) \right),$$

where the functions $F_j(\zeta, \nu)$, $j = 1, 2$ are given in (79). As with the Bessel functions, we can differentiate this expansion and obtain

$$(86) \quad H_\nu^{(1)'}(\nu z) \sim -2e^{-i\pi/3} \psi(\zeta) \left(\frac{Ai_{-1}(\nu^{2/3}\zeta)}{\nu^{4/3}} G_1(\zeta, \nu) + \frac{Ai'_{-1}(\nu^{2/3}\zeta)}{\nu^{2/3}} G_2(\zeta, \nu) \right).$$

The functions $G_j(\zeta, \nu)$, $j = 1, 2$, are given in (80). As above, these expansions are uniform in $|\arg z| \leq \pi - \epsilon$, for any $\epsilon > 0$.

6.3. Asymptotics for Airy Functions. The asymptotics of the Bessel and Hankel functions are obtained from the asymptotic expansions for the Airy functions (e.g. [15, appendix]). These expansions depend on whether the argument is near the zeros of the Airy functions along the negative real axis, or away from them. For the Hankel functions of the first kind we only need the asymptotics away from the zeros of the Airy functions. Let $\xi = (2/3)w^{3/2}$ where ξ is the principle value. For $|\arg w| < \pi - \epsilon$, for any $\epsilon > 0$, the argument w is away from the zeros of the Airy function, and we have

$$(87) \quad Ai(w) \sim \frac{e^{-\xi}}{2\pi^{1/2}w^{1/4}} \hat{F}_1(\xi), \quad \text{and} \quad Ai'(w) \sim \frac{-w^{1/4}e^{-\xi}}{2\pi^{1/2}} \hat{G}_1(\xi).$$

The functions appearing in these expansions are

$$(88) \quad \hat{F}_1(\xi) \equiv \sum_{j=0}^{\infty} \frac{c_j}{\xi^j}, \quad \text{and} \quad \hat{G}_1(\xi) \equiv \sum_{j=0}^{\infty} \frac{d_j}{\xi^j}.$$

The coefficients c_j and d_j are numerical constants given by

$$(89) \quad c_j = (-1)^j \frac{(2j+1)(2j+3)\cdots(6j-1)}{j!(216)^j},$$

and

$$(90) \quad d_j = c_j + c_{j-1}(j-5/6) = -\frac{6j+1}{6j-1}c_j,$$

with $c_0 = 1$ so that $d_0 = 1$. We note that the coefficients (c_j, d_j) are related to the coefficients (u_j, v_j) used by Olver as $c_j = (-1)^j u_j$ and $d_j = (-1)^j v_j$.

6.4. Uniform Asymptotics near the Eye-Shaped Region K . Of particular importance is the application of the uniform asymptotics that follow from (83)-(87) for z near the eye-shaped region K defined in section 5.2. Recall that for a fixed $\epsilon > 0$, we define a neighborhood $\Omega_{1,\epsilon}$ of K to be all $z \in \mathbb{C}^+$ so that $\text{dist}(z, \partial K^+) < \epsilon$ excluding small neighborhoods of ± 1 so that $z \in \Omega_{1,\epsilon}$ satisfies $|z + 1| > \epsilon$ and $|z - 1| > \epsilon$.

For $z \in \Omega_{1,\epsilon}$, the uniform asymptotics of the Bessel functions and their derivatives follow from (83) and the estimates on the Airy function and its derivative away from their zeros and given in (87). The uniform expansion of $J_\nu(\nu z)$ and its derivative involve the series $F_1(\zeta)$, $F_2(\zeta)$, given in (79), and $\hat{F}_1(\xi)$ and $\hat{G}_1(\xi)$, defined in (88). For the Bessel functions, we use $\xi = \nu\rho$ in the expansion (87). We obtain:

$$(91) \quad \begin{aligned} J_\nu(\nu z) &= \frac{\phi(\zeta)e^{-\nu\rho}}{2\pi^{1/2}\nu^{1/2}\zeta^{1/4}} \left\{ 1 + \frac{1}{\nu} \left(\frac{c_1}{\rho} - \zeta^{1/2}B_0(\zeta) \right) \right. \\ &\quad \left. + \frac{1}{\nu^2} \left(A_1(\zeta) + \frac{c_2}{\rho^2} - \frac{\zeta^{1/2}B_0(\zeta)d_1}{\rho} \right) + \mathcal{O}\left(\frac{1}{\nu^3}\right) \right\}, \end{aligned}$$

and for the derivative,

$$(92) \quad \begin{aligned} J'_\nu(\nu z) &= -\frac{\psi(\zeta)e^{-\nu\rho}}{2\pi^{1/2}\nu^{1/2}\zeta^{1/4}} \left\{ -\zeta^{1/2} + \frac{1}{\nu} \left(C_0(\zeta) - \frac{\zeta^{1/2}d_1}{\rho} \right) \right. \\ &\quad \left. + \frac{1}{\nu^2} \left(\frac{C_0(\zeta)c_1}{\rho} - \frac{\zeta^{1/2}d_2}{\rho^2} - \zeta^{1/2}D_1(\zeta) \right) + \mathcal{O}\left(\frac{1}{\nu^3}\right) \right\}, \end{aligned}$$

where the coefficients $C_j(\xi)$ and $D_j(\xi)$ are defined in (81) and (82).

We also need the asymptotics for the Hankel functions of the first kind and their derivatives for $z \in \Omega_{1,\epsilon}$. These are obtained from (85) using the expansions of the Airy functions. We note that for $z \in \Omega_{1,\epsilon}$, it follows from the map $z \rightarrow \zeta$ that $\arg \zeta \sim -\pi/3$. Consequently, for $H_\nu^{(1)}(\nu z)$, we have $\arg(e^{2\pi i/3}\zeta) \sim \pi/3$, and we need the asymptotics of the Airy function away from its zeros given in (87). Using these asymptotics (87), we find for the Hankel functions of the first kind, with $\xi = -\nu\rho$

$$(93) \quad \begin{aligned} H_\nu^{(1)}(\nu z) &= \frac{-i\phi(\zeta)e^{\nu\rho}}{\pi^{1/2}\zeta^{1/4}\nu^{1/2}} \left\{ 1 + \frac{1}{\nu} \left(\zeta^{1/2}B_0(\zeta) - \frac{c_1}{\rho} \right) \right. \\ &\quad \left. + \frac{1}{\nu^2} \left(A_1(\zeta) + \zeta^{1/2}B_1(\zeta) + \frac{c_2}{\rho^2} - \frac{\zeta^{1/2}d_1B_0(\zeta)}{\rho} \right) + \mathcal{O}\left(\frac{1}{\nu^3}\right) \right\}. \end{aligned}$$

As for the derivative, we use the same expansions of the Airy functions in (86), and obtain

$$(94) \quad \begin{aligned} H_\nu^{(1)'}(\nu z) &= \frac{i\psi(\zeta)e^{\nu\rho}}{\pi^{1/2}\zeta^{1/4}\nu^{1/2}} \left\{ \zeta^{1/2} + \frac{1}{\nu} \left(C_0(\zeta) - \frac{d_1\zeta^{1/2}}{\rho} \right) \right. \\ &\quad \left. + \frac{1}{\nu^2} \left(-\frac{C_0(\zeta)c_1}{\rho} + \frac{\zeta^{1/2}d_2}{\rho^2} + \zeta^{1/2}D_1(\zeta) \right) + \mathcal{O}\left(\frac{1}{\nu^3}\right) \right\}. \end{aligned}$$

6.5. Auxiliary Asymptotic Expansions. The variable $\tilde{z}(z) = [z^2 - V_0/\nu^2]^{1/2}$ carries the information about the perturbing potential with strength V_0 . We need the expansions as $\nu \rightarrow \infty$ of various quantities depending on \tilde{z} for z in a fixed set $\Omega_{1,\epsilon}$ for which z is bounded away from ± 1 and 0. First, we note that

$$(95) \quad \tilde{z}(z) = z - \frac{V_0}{2\nu^2 z} + \mathcal{O}\left(\frac{1}{\nu^4}\right).$$

Recalling the definition of ρ in (54), we find

$$(96) \quad \rho(\tilde{z}) = \rho(z) + \frac{V_0}{2\nu^2} \left[\frac{(1-z^2)^{1/2}}{z^2} \right] + \mathcal{O}\left(\frac{1}{\nu^4}\right).$$

It follows from the definition of ζ in (54) that

$$(97) \quad \begin{aligned} \frac{\zeta(\tilde{z})}{\zeta(z)} &= 1 + \frac{V_0}{3\nu^2} \left[\frac{\rho'}{z\rho} \right] + \mathcal{O}\left(\frac{1}{\nu^4}\right) \\ &= 1 + \frac{V_0}{2\nu^2 \zeta^{3/2}} \left[\frac{(1-z^2)^{1/2}}{z^2} \right] + \mathcal{O}\left(\frac{1}{\nu^4}\right). \end{aligned}$$

We write $\tilde{\rho}$ and $\tilde{\zeta}$ for $\rho(\tilde{z})$ and $\zeta(\tilde{z})$. We need the expansion of the following combination that follows from (96)–(97):

$$(98) \quad \frac{\phi(\tilde{\zeta})}{\phi(\zeta)} \left(\frac{\zeta}{\tilde{\zeta}} \right)^{1/4} = 1 - \frac{V_0}{4\nu^2(1-z^2)} + \mathcal{O}\left(\frac{1}{\nu^4}\right).$$

Finally, we need an asymptotic expansion for the exponentials appearing in the products of Bessel functions, see (91)–(92). This follows from (96):

$$(99) \quad e^{-\nu(\tilde{\rho}-\rho)} = 1 - \frac{V_0(1-z^2)^{1/2}}{2\nu z^2} + \frac{V_0^2(1-z^2)}{8\nu^2 z^4} + \mathcal{O}\left(\frac{1}{\nu^3}\right).$$

6.6. Condition for Zeros on the m^{th} -Sheet. We now compute the asymptotic expansion of $F_0^{(\nu)}(\nu z)$, defined in (56), to order ν^{-2} , and of $G_0^{(\nu)}(\nu z)$, defined in (57), to order ν^{-3} , using the uniform asymptotics of section 6.4. As for $F_0^{(\nu)}(\nu z)$, recall that

$$(100) \quad F_0^{(\nu)}(\nu z) = \nu \tilde{z}(z) J'_\nu(\nu \tilde{z}(z)) H_\nu^{(1)}(\nu z) - \nu z J_\nu(\nu \tilde{z}(z)) H_\nu^{(1)'}(\nu z).$$

From the asymptotics of the Bessel functions (91)–(92) and of the Hankel functions of the first kind (93)–(94), together with auxiliary expansions (96)–(97), we find that the first term on the right in (100) has the expansion in inverse powers of ν :

$$(101) \quad \begin{aligned} &\nu \tilde{z}(z) J'_\nu(\nu \tilde{z}(z)) H_\nu^{(1)}(\nu z) \\ &= \frac{i}{\pi} \left\{ -1 + \frac{1}{\nu} \left[\frac{V_0(1-z^2)^{1/2}}{2z^2} + \frac{C_0(\zeta)}{\zeta^{1/2}} + \frac{c_1 - d_1}{\rho} - \zeta^{1/2} B_0(\zeta) \right] + \mathcal{O}\left(\frac{1}{\nu^2}\right) \right\}. \end{aligned}$$

Similarly, we find for the second term on the right in (100), we the expansion in ν is

$$\begin{aligned} & \nu z J_\nu(\nu \tilde{z}(z)) H_\nu^{(1)'}(\nu z) \\ &= \frac{i}{\pi} \left\{ 1 + \frac{1}{\nu} \left[\frac{-V_0(1-z^2)^{1/2}}{2z^2} + \frac{C_0(\zeta)}{\zeta^{1/2}} - \zeta^{1/2} B_0(\zeta) + \frac{c_1 - d_1}{\rho} \right] + \mathcal{O}\left(\frac{1}{\nu^2}\right) \right\}. \end{aligned} \quad (102)$$

Subtracting the expansion (102) from (101), we find that $F_0^{(\nu)}(\nu z)$ has the asymptotic form

$$(103) \quad F_0^{(\nu)}(\nu z) = \frac{-2i}{\pi} \left\{ 1 - \frac{1}{\nu} \left[\frac{V_0(1-z^2)^{1/2}}{2z^2} \right] + \mathcal{O}\left(\frac{1}{\nu^2}\right) \right\}.$$

This is (60). Note that this implies that, asymptotically, there are no zeros on the physical sheet, as expected.

We turn to the second quantity $G_0^{(\nu)}(\nu z)$ defined in (57):

$$(104) \quad G_0^{(\nu)}(\nu z) = \nu \tilde{z}(z) J_\nu'(\nu \tilde{z}) J_\nu(\nu z) - \nu z J_\nu(\nu \tilde{z}) J_\nu'(\nu z).$$

We treat each term separately. For the first term, we find the following asymptotic expansion in ν :

$$\begin{aligned} & \nu \tilde{z}(z) J_\nu'(\nu \tilde{z}) J_\nu(\nu z) \\ &= \left(\frac{-1}{2\pi} \right) e^{-\nu(\rho+\bar{\rho})} \left\{ -1 + \frac{1}{\nu} \left[\frac{C_0(\zeta)}{\zeta^{1/2}} - \frac{c_1 + d_1}{\rho} + \zeta^{1/2} B_0(\zeta) \right] \right. \\ & \quad + \frac{1}{\nu^2} \left[\frac{2c_1 C_0(\zeta)}{\rho \zeta^{1/2}} - \frac{d_1 c_1 + d_2 + c_2}{\rho^2} - (B_0(\zeta) C_0(\zeta) + A_1(\zeta) + D_1(\zeta)) + \frac{2\zeta^{1/2} B_0(\zeta) d_1}{\rho} \right. \\ & \quad \left. \left. - \frac{V_0}{4(1-z^2)} \right] + \mathcal{O}\left(\frac{1}{\nu^3}\right) \right\}. \end{aligned} \quad (105)$$

For the second term in $G_0^{(\nu)}(\nu z)$, we obtain

$$\begin{aligned} & \nu z J_\nu(\nu \tilde{z}) J_\nu'(\nu z) \\ &= \left(\frac{-1}{2\pi} \right) e^{-\nu(\rho+\bar{\rho})} \left\{ -1 + \frac{1}{\nu} \left[\frac{C_0(\zeta)}{\zeta^{1/2}} - \frac{c_1 + d_1}{\rho} + \zeta^{1/2} B_0(\zeta) \right] \right. \\ & \quad + \frac{1}{\nu^2} \left[\frac{2c_1 C_0(\zeta)}{\rho \zeta^{1/2}} - \frac{d_1 c_1 + d_2 + c_2}{\rho^2} - (B_0(\zeta) C_0(\zeta) + A_1(\zeta) + D_1(\zeta)) + \frac{2\zeta^{1/2} B_0(\zeta) d_1}{\rho} \right. \\ & \quad \left. \left. + \frac{V_0}{4(1-z^2)} \right] + \mathcal{O}\left(\frac{1}{\nu^3}\right) \right\}. \end{aligned} \quad (106)$$

Subtracting (106) from (105), we obtain the expansion for $G_0^{(\nu)}(\nu z)$:

$$(107) \quad G_0^{(\nu)}(\nu z) = \frac{e^{-\nu(\rho+\bar{\rho})}}{2\pi} \left[\frac{V_0}{2\nu^2(1-z^2)} + \mathcal{O}\left(\frac{1}{\nu^3}\right) \right] = \frac{e^{-2\nu\rho}}{2\pi} \left[\frac{V_0}{2\nu^2(1-z^2)} + \mathcal{O}\left(\frac{1}{\nu^3}\right) \right].$$

Note that when $V_0 = 0$, this term vanishes to higher order in ν . This establishes (61).

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