SOME REMARKS ON SCATTERING RESONANCES IN EVEN DIMENSIONAL EUCLIDEAN SCATTERING

T. J. CHRISTIANSEN AND P. D. HISLOP

Abstract. The purpose of this paper is to prove some results about quantum mechanical black box scattering in even dimensions $d \geq 2$. We study the scattering matrix and prove some identities which hold for its meromorphic continuation onto Λ, the Riemann surface of the logarithm function. We study the multiplicities of the poles of the continued scattering matrix on each sheet of Λ and relate these to the multiplicities of the poles of the resolvent on each sheet. Moreover, we show that the poles of the scattering matrix on the mth sheet of Λ are related to the zeros of a scalar function defined on the physical sheet. This paper contains a number of results about "pure imaginary" resonances. As an example, in contrast with the odd-dimensional case, we show that there are no "purely imaginary" resonances on any sheet of Λ for Schrödinger operators with potentials $0 \leq V \in L^\infty_0(\mathbb{R}^d)$ in even dimensions.

1. Introduction

This paper presents several results about resonances in quantum mechanical black box Euclidean scattering in even dimensions. There are several objects which naturally may be called resonances. *Resolvent resonances* occur as poles of the meromorphic continuation of the cut-off resolvent, while *scattering resonances* are poles of the meromorphic continuation of the scattering matrix. In this setting both lie on Λ, the logarithmic cover of $\mathbb{C} \setminus \{0\}$. We prove an identity clarifying the relationship between these two. Moreover, we show there is a scalar function on the physical region, the zeros of which correspond to poles of the scattering matrix on the mth sheet of Λ. We show the absence of "purely imaginary" resonances for certain classes of operators. This extends results of Beale [2], and is in sharp contrast with the odd-dimensional case. We observe a small correction to an oft-quoted identity of [26] about symmetries of the scattering matrix. This correction is important in the context of "pure imaginary" resonances in even dimensions.

Let $V \in L^\infty_0(\mathbb{R}^d; \mathbb{C})$, and let $\Delta \leq 0$ be the Laplacian on $\mathbb{R}^d$. Set $P = -\Delta + V$, and, for $\operatorname{Im} \lambda > 0$, set $R(\lambda) = (P - \lambda^2)^{-1}$ be the resolvent, which is bounded as an operator on $L^2(\mathbb{R}^d)$ for all but finitely many λ with $\operatorname{Im} \lambda > 0$. Let $\chi \in L^\infty_0(\mathbb{R}^d)$ be one on the support of V. It is well known that $\chi R(\lambda) \chi$ has a meromorphic extension to $\mathbb{C}$ if d is

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odd. If d is even, the extension is to Λ, the logarithmic cover of $\mathbb{C}\setminus\{0\}$. In the latter case, we identify the physical space, where $R(\lambda)$ is bounded on L^2 (except for finitely many points), with the subset of Λ defined by $\Lambda_0 \stackrel{\text{def}}{=} \{\lambda \in \Lambda : 0 < \arg\lambda < \pi\}$. The poles of the meromorphic continuation of $\chi R(\lambda)\chi$ are called *(resolvent) resonances*. Moreover, a similar extension and a similar definition can be made for many compactly supported perturbations of $-\Delta$ on $\mathbb{R}^d$; see the "black box" definition of Sjöstrand-Zworski [27], recalled here in Section 2. For example, the class of operators for which one can make this meromorphic continuation and the subsequent definition of resonances includes the Dirichlet or Neumann Laplacian on $\mathbb{R}^d\setminus\overline{\mathcal{O}}$, where $\mathcal{O}\subset\mathbb{R}^d$ is a bounded open set with smooth boundary.

We shall be particularly interested in the case of even d. For $m\in\mathbb{Z}$, we set

$$\Lambda_m = \{\lambda\in\Lambda : m\pi < \arg\lambda < (m+1)\pi\}.$$

Thus, with our convention, Λ_0 corresponds to the physical region in even dimensional scattering. We call a point $\lambda\in\Lambda$ "pure imaginary" if $\arg\lambda = \pi/2 + k\pi$ for some $k\in\mathbb{Z}$. An example of our results is the following.

Theorem 1.1. *Let the dimension $d\geq 2$ be even. Let $V\in L_0^\infty(\mathbb{R}^d;\mathbb{R})$ be a potential with fixed sign, that is either $V\geq 0$ or $-V\geq 0$. If $V\geq 0$, the nonnegative Schrödinger operator $H_V=-\Delta+V$ has no purely imaginary (resolvent) resonances on any sheet Λ_m, $m\in\mathbb{Z}$. If $-V\geq 0$, suppose that the operator H_V has $0\leq N_V<\infty$ negative eigenvalues on Λ_0. Then the lower-semi-bounded operator H_V has at most N_V purely imaginary (resolvent) resonances on each sheet Λ_m, for $m\in\mathbb{Z}\setminus\{0\}$.*

This theorem is proved in Section 6. We note that this is in sharp contrast with the odd-dimensional case. In odd dimensions, Lax and Phillips [13] proved lower bounds on the number of purely imaginary resonances for Dirichlet or Neumann obstacle scattering. They noted that their technique applies to Schrödinger operators with strictly nonnegative compactly supported potentials. See also [17] for a related result. This was extended by A. Vasy to compactly-supported, bounded potentials of fixed sign [31]. To be more precise, let $V\in L_0^\infty(\mathbb{R}^d)$ be a fixed sign potential so that there is an $\epsilon>0$, and some nontrivial ball $B\subset\mathbb{R}^d$ so that $|V|\geq\epsilon\chi_B$, where χ_B is the characteristic function of the ball B. Then [13, 31] showed that in *odd dimensions* $d\geq 3$, the Schrödinger operator $H_V=-\Delta+V$ has an infinite number of purely imaginary resonances on the nonphysical sheet. In fact, they proved a qualitative lower bound. The number of such poles $N_{\text{im}}(r)$ with norm at most r satisfies, for large r:

$$N_{\text{im}}(r)\geq c_V r^{d-1} \tag{1.1}$$

for a positive constant c_V. Here and always the resonances are counted with multiplicity.

The results of Lax and Phillips [13] for obstacle scattering were extended to certain Robin-type boundary conditions by Beale [2]. Beale also noted that for even-dimensional scattering there are no purely imaginary resonances on Λ_{-1} (and hence on Λ_1) for Dirichlet or Neumann obstacle scattering, and at most finitely many for certain Robin-type boundary conditions. See Corollary 5.3 for a more precise statement in the even-dimensional case and for our extension of these results. We prove additional results on the absence of purely imaginary resonances in Section 5.

We begin this paper with Proposition 2.1, a somewhat subtle correction to an identity from [26]. We include this because we are unaware of a reference in which the correct version is explicitly stated and because the subtle distinction has important consequences for the existence or not of purely imaginary resonances in even dimensions. In fact the correct version has been implicitly mentioned in [13, 2] in the context of purely imaginary resonances.

Another result of our note is Proposition 3.5. This proposition tells us that in even dimensions to study the poles of the scattering matrix on Λ_m, it suffices to study the zeros of a function which is holomorphic on Λ_0 with the possible exception of at most finitely many poles there. This function is related in an explicit way to the scattering matrix. This is familiar in the odd dimensional case, where the poles of the scattering matrix in the nonphysical half plane are, with perhaps finitely many exceptions, determined by the zeros of the determinant of the scattering matrix in the physical half plane.

Theorem 4.5 and its Corollary 4.9 give a relationship between the poles of the resolvent ("resolvent resonances") and the poles of the scattering matrix ("scattering resonances"). Again, this relation is well known in the odd dimensional case and is known for a very limited subset of Λ in the even dimensional case– see Section 4 for references. To the best of our knowledge there is not a proof of this result in the literature which is valid for all points in Λ.

There are a number of results on the distribution of resonances which are not intimately tied to the parity of the dimension. At least some of these rely on complex scaling, and as a consequence can only say something about resonances "near" the physical half plane. "Near" generally meaning in some sector, of opening no greater than $\pi/2$. We make no attempt to survey such results, but merely mention as an example results on the distribution of resonances "near" the physical plane for the Laplacian on the exterior of a strictly convex obstacle, [28].

For questions about distribution of resonances further from the physical half-plane, the case of even dimensions has received far less attention than the case of odd dimensions. Exceptions include [10, 32, 33] in which upper bounds on resonance-counting functions are obtained, and [4, 5, 23, 24, 30] for lower bounds. Two papers which focus on resonances on the sheets $\Lambda_{\pm 1}$ are [2] and [35]. Results of Beale [2] for purely imaginary resonances are recalled in Section 5. The paper [35] proves a Poisson formula in even dimensions.

2. The "black box" formalism and relations for the scattering matrix

In this section, we allow $d \geq 2$ to be either even or odd. Here we assume P is a compactly supported "black box" perturbation of the Laplacian on $\mathbb{R}^d$ satisfying the conditions of [27], including that P is self-adjoint. We remark that some of the results of this paper use the self-adjointness of P in an essential way.

We recall the black box assumptions below for the convenience of the reader. Note that if $\mathcal{O} \subset \mathbb{R}^d$ is a bounded open set with smooth boundary $\partial\mathcal{O}$, and $V \in L^\infty_0(\mathbb{R}^d \setminus \mathcal{O};\mathbb{R})$, these hypothesis are satisfied by the operator $-\Delta + V$ on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ with Dirichlet or Neumann boundary conditions on $\partial\mathcal{O}$.

In recalling the assumptions of [27] we use similar notation. By a black box operator we mean an operator P defined on a domain $\mathcal{D} \subset \mathcal{H}$ satisfying the conditions below. Let $U \subset \mathbb{R}^d$ be a bounded connected open set. Let $\mathcal{H}$ be a complex Hilbert space with orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_U \oplus L^2(\mathbb{R}^d \setminus U).$$

Following [27], we denote the corresponding orthogonal projections by $u \mapsto u_{\restriction U}$ and $u \mapsto u_{\restriction \mathbb{R}^d\setminus U}$. We assume that the operator $P : \mathcal{H} \to \mathcal{H}$ is semibounded below, self-adjoint with domain $\mathcal{D} \subset \mathcal{H}$. Furthermore, if $u \in H^2(\mathbb{R}^d \setminus U)$ and u vanishes near U, then $u \in \mathcal{D}$; and conversely $\mathcal{D}_{|\mathbb{R}^d\setminus U} \subset H^2(\mathbb{R}^d \setminus U)$. The operator P is $-\Delta$ outside U:

$$Pu_{|\mathbb{R}^d\setminus U} = -\Delta u_{|\mathbb{R}^d\setminus U} \text{ for all } u \in \mathcal{D}$$

and

$$\mathbf{1}_U (P + i)^{-1} \text{ is compact}$$

where $\mathbf{1}_U$ is the characteristic function of U; that is, projection onto $\mathcal{H}_U$.

Let $\chi \in L^\infty_0(\mathbb{R}^d)$. Under these conditions on P, the cut off resolvent $\chi(P - \lambda^2)^{-1}\chi$ defined on the physical sheet $0 < \arg\lambda < \pi$ has a meromorphic continuation to $\mathbb{C}$ if d is odd and to Λ if d is even. While this is well-known for specific operators, in this generality we refer the reader to, for example, [27, Theorem 1.1] or the proof of Proposition 4.1 of [22].

For future reference, we note that we shall use the notation

$$\langle x\rangle^s\mathcal{H} \stackrel{\text{def}}{=} \mathcal{H}_U \oplus \langle x\rangle^s L^2(\mathbb{R}^d \setminus U)$$

and similarly for $\langle x\rangle^s\mathcal{D} \subset \langle x\rangle^s\mathcal{H}$. Here $\langle x\rangle = (1 + |x|^2)^{1/2}$.

We work with the scattering matrix $S(\lambda)$ associated to P; one explicit expression for it is recalled in (2.10) and another in Proposition 4.1. On the positive real axis $\{\arg\lambda = 0\}$ it is unitary, and it differs from the identity by a trace class operator.

Moreover, it has a meromorphic continuation to the complex plane (for d odd) or to Λ (for d even), as follows from the meromorphic continuation of the cut-off resolvent and the expression for the scattering matrix recalled in Proposition 4.1 (see also [26, 15]).

Following [26] we write

$$\overline{\lambda} = |\lambda| \exp(-i \arg \lambda) \text{ for } \lambda \in \Lambda.$$

For $\lambda \in \Lambda$ the complex involution $\lambda \mapsto \overline{\lambda}$ takes Λ_m to Λ_{-m-1}. The unitarity of the scattering matrix for $\arg \lambda = 0$ means that

$$S^*(\overline{\lambda})S(\lambda) = I$$

in any dimension.

In [26, Theorem 1], the following identity is stated for the scattering matrix $S(\lambda)$ for a combination of an obstacle and potential perturbation of the Laplacian on $\mathbb{R}^d$:

$$S(\overline{\lambda})^* = \begin{cases} S(-\lambda) & \text{if } d \text{ is odd} \\ 2I - S(e^{i\pi}\lambda) & \text{if } d \text{ is even.} \end{cases} \tag{2.1}$$

These relations have been widely repeated by others, including the present authors. However, it appears that there is a slight error in the identity. In most or all of the cases where (2.1) rather than (2.2) has been stated, the difference between the two versions is unimportant to the subsequent discussion. We shall see in Section 5 that the difference is important to results on pure imaginary resonances, and that is why we include Proposition 2.1 here.

Define $\mathcal{R} : C^\infty(\mathbb{S}^{d-1}) \to C^\infty(\mathbb{S}^{d-1})$ by $(\mathcal{R}f)(\theta) = f(-\theta)$. We shall use the same notation for the continuous extension of $\mathcal{R}$ to $L^2(\mathbb{S}^{d-1})$.

Proposition 2.1. *For P satisfying the black box conditions, the scattering matrix $S(\lambda)$ satisfies*

$$S(\overline{\lambda})^* = \begin{cases} \mathcal{R}S(-\lambda)\mathcal{R} & \textit{if } d \textit{ is odd} \\ 2I - \mathcal{R}S(e^{i\pi}\lambda)\mathcal{R} & \textit{if } d \textit{ is even.} \end{cases} \tag{2.2}$$

There are certainly many instances in the literature which are consistent with (2.2) rather than (2.1). These include, for example, [34] and the works [13] and [2], both of the latter related to work in this paper, see Corollary 5.3. However, since we are unaware of an explicit reference for (2.2) and because the distinction between the two versions is important for our results, we include Proposition 2.1 and a proof here.

We note that these equalities show that for the operators P we consider, in even dimensions d, if $\lambda_0 \in \Lambda$ is a pole of $S(\lambda)$ with $m\pi < \arg \lambda_0 < (m+1)\pi$, then $\overline{\lambda_0}\, e^{i\pi}$ is a pole of $S(\lambda)$, and $-m\pi < \arg(\overline{\lambda_0}\, e^{i\pi}) < (-m+1)\pi$. Thus, poles of P on Λ_m are symmetric with poles on Λ_{-m}. This replaces the symmetry relation for the case of odd-dimensional d, which is more familiar: for odd d, $\lambda_0 \in \mathbb{C}$ is a pole of the scattering matrix if and only if $-\overline{\lambda}_0$ is a pole of the scattering matrix.

In order to prove the proposition, we recall some background. Let $g \in \mathcal{H}$ satisfy $g_{\restriction |x|>R} = 0$ for some finite R. Then for $\lambda \in \mathbb{R} \setminus \{0\}$ there are unique $u_\pm \in \langle x \rangle^{1/2+\epsilon}\mathcal{D}$ satisfying

$$(P-\lambda^2)u_\pm = g \tag{2.3}$$

$$u_\pm(x) = e^{\pm i\lambda|x|}|x|^{-(d-1)/2}(\alpha_\pm(x/|x|)+o(1)) \text{ as } |x|\to\infty \tag{2.4}$$

for some functions $\alpha_\pm \in C^\infty(\mathbb{S}^{d-1})$.

Let $\omega \in \mathbb{S}^{d-1}$ and let $\psi \in C^\infty(\mathbb{R}^d)$ be 0 on U and satisfy $1-\psi \in C_c^\infty(\mathbb{R}^d)$. Applying the results recalled above, we see that there are unique $v_\pm(x,k,\omega) \in \langle x \rangle^{1/2+\epsilon}\mathcal{D}$ and thus unique scattering amplitudes $s_\pm(\theta,\lambda,\omega)$ which satisfy (Cf. [26, Section 2]; we use similar notation here.)

$$(P-\lambda^2)\left[v_\pm(x,\lambda,\omega)+\psi(x)e^{i\lambda x\cdot\omega}\right] = 0 \tag{2.5}$$

with

$$v_\pm(r\theta,\lambda,\omega) = e^{\pm i\lambda r}r^{(1-d)/2}\left[s_\pm(\theta,\lambda,\omega)+o(1)\right] \text{ as } r\to\infty. \tag{2.6}$$

Now [26, Lemma 2.1] states

$$\overline{s}_+(\theta,\lambda,\omega) = s_-(\theta,\overline{\lambda},-\omega) \tag{2.7}$$

$$s_+(\theta,e^{i\pi}\lambda,\omega) = s_-(\theta,\lambda,-\omega) \tag{2.8}$$

$$s_-(\theta,\lambda,\omega) = s_-(\omega,\lambda,\theta). \tag{2.9}$$

The reader should note that we mean by the notation $\overline{s}$ the usual complex conjugate on $\mathbb{C}$. While this is possibly confusing, an alternate notation with *, risks being confused with the adjoint of an operator.

Strictly speaking, [26] proved (2.7) -(2.9) only for P which are $-\Delta+V$ in the exterior of a smooth, compact obstacle with Dirichlet or Neumann boundary conditions. However, it is not difficult to see that their proof extends to self-adjoint P satisfying the black box type conditions we consider here.

The following lemma is analogous to [26, 2.7]. It seems that it is this identity in which [26] made an error.

Lemma 2.2. *With the notation as above, $s_+(\theta,\lambda,\omega) = s_+(-\omega,\lambda,-\theta)$.*

Proof. We have

$$s_+(\theta,\lambda,\omega) = \overline{s}_-(\theta,\overline{\lambda},-\omega) = \overline{s}_-(-\omega,\overline{\lambda},\theta) = s_+(-\omega,\lambda,-\theta)$$

where we have used respectively (2.7), (2.9), and (2.7). □

In [26, (2.7)] it is claimed that $s_+(\theta,\lambda,\omega) = s_+(-\theta,\lambda,-\omega)$. This is in general not true, as we shall see. Here we denote by $\tilde{\mathcal{R}}$ the operator which on $L^2(\mathbb{R}^d)$ is given by $(\tilde{\mathcal{R}}f)(x) = f(-x)$. We shall also denote by $\tilde{\mathcal{R}}\mathcal{O}$ the set $\{x \in \mathbb{R}^d : -x \in \mathcal{O}\}$, with a similar definition of $\tilde{\mathcal{R}}\partial\mathcal{O}$.

Lemma 2.3. *Let $\mathcal{O} \subset \mathbb{R}^d$ be an open bounded set so that $\mathcal{O}$ has smooth boundary $\partial\mathcal{O}$, and let $V \in L^\infty_0(\mathbb{R}^d;\mathbb{R})$. Let P denote $-\Delta + V$ on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ with Dirichlet (Neumann) boundary conditions. Let $P\tilde{\mathcal{R}}$ denote $-\Delta + V(-x)$ on $\mathbb{R}^d \setminus (\tilde{\mathcal{R}}\overline{\mathcal{O}})$ with Dirichlet (respectively, Neumann) boundary condition. Then*

$$s_{+,P\tilde{\mathcal{R}}}(\theta,\lambda,\omega) = s_{+,P}(-\theta,\lambda,-\omega)$$

where $s_{+,P\tilde{\mathcal{R}}}$ and $s_{+,P}$ denote the functions s_+ corresponding to $P\tilde{\mathcal{R}}$ and P, respectively.

We understand in the statement of this lemma that we may take $\mathcal{O} = \emptyset$, in which case there is no boundary condition.

Proof. We use notation similar to that of (2.5). We shall add a subscript P to v_+ writing $v_{+,P}$ to denote its dependence on P.

Note that $v_{+,P}(-x,\lambda,-\omega)$ satisfies

$$(P\tilde{\mathcal{R}} - \lambda^2)\left[v_{+,P}(-x,\lambda,-\omega) + \psi e^{i\lambda x\cdot\omega}\right] = 0 \text{ in } \mathbb{R}^d \setminus \tilde{\mathcal{R}}\overline{\mathcal{O}}$$

with $v_{+,P}(-x)$ satisfying the Dirichlet (or Neumann) boundary condition on $\partial\mathcal{R}\mathcal{O}$. Moreover, since $v_{+,P}(-x,\lambda,-\omega)$ satisfies a radiation condition as in (2.4) with the $+$ sign, we have for $\lambda \in \mathbb{R}\setminus\{0\}$ that $v_{+,P\mathcal{R}}(x,\lambda,\omega) = v_{+,P}(-x,\lambda,-\omega)$ and thus

$$s_{+,P\tilde{\mathcal{R}}}(\theta,\lambda,\omega) = s_{+,P}(-\theta,\lambda,-\omega).$$

□

We continue with the notation of the previous lemma, and show that [26, (2.7)], that is, $s_+(\theta,\lambda,\omega) = s_+(-\theta,\lambda,-\omega)$, cannot hold in general. The scattering matrix at energy λ is determined by $s_+(\theta,\lambda,\omega)$, see (2.10). Thus by uniqueness results of inverse scattering theory if either $V \equiv 0$ and $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ is connected (e.g. [14, Theorem 5.6]) *or* $\mathcal{O} = \emptyset$ and $V \in C_c(\mathbb{R}^d;\mathbb{R})$ (e.g. [6, 7, 25]), *and* if $s_{+,P\tilde{\mathcal{R}}}(\theta,\lambda,\omega) = s_{+,P}(\theta,\lambda,\omega)$, for all $\lambda \in (0,\infty)$ and all $\theta,\omega \in \mathbb{S}^{d-1}$, then $P\tilde{\mathcal{R}} = P$ and $\tilde{\mathcal{R}}\mathcal{O} = \mathcal{O}$. Thus if we use Lemma 2.2 we see that [26, (2.7)] is not true in general. However, we have shown, if we temporarily assume Proposition 2.1, the following corollary.

Corollary 2.4. *Suppose $\tilde{\mathcal{R}}\mathcal{O} = \mathcal{O}$ and $V(-x) \equiv V(x)$, with $\mathcal{O}$ and V satisfying the conditions of Lemma 2.3. In this case, if d is odd, $S(\overline{\lambda})^* = S(-\lambda)$, and if d is even, $S(\overline{\lambda})^* = 2I - S(e^{i\pi}\lambda)$.*

Now we prove the proposition. We return to omitting the subscript P on s_+, as we will be working with a fixed operator P.

Proof of Proposition 2.1. We have [26, Section 2]

$$S(\lambda)h(\theta) = h(\theta) + \left(\frac{i\lambda}{2\pi}\right)^{(d-1)/2} \int h(\omega)\overline{s}_-(-\theta,\overline{\lambda},\omega)dS_\omega. \tag{2.10}$$

Now let $\lambda > 0$, that is, $\arg \lambda = 0$. From (2.10), we see that for such λ the kernel of $S^*(\overline{\lambda}) - I$ is given by

$$\left(\frac{\lambda}{2\pi}\right)^{(d-1)/2} e^{-\pi i(d-1)/4} s_-(-\omega, \lambda, \theta). \tag{2.11}$$

On the other hand, the kernel of $S(e^{i\pi}\lambda) - I$ is

$$\begin{aligned} \left(\frac{ie^{i\pi}\lambda}{2\pi}\right)^{(d-1)/2} \overline{s}_-(-\theta, e^{-i\pi}\overline{\lambda}, \omega) &= \left(\frac{\lambda}{2\pi}\right)^{(d-1)/2} e^{3\pi i(d-1)/4} s_+(-\theta, e^{i\pi}\lambda, -\omega) \\ &= \left(\frac{\lambda}{2\pi}\right)^{(d-1)/2} e^{3\pi i(d-1)/4} s_-(-\theta, \lambda, \omega) \end{aligned} \tag{2.12}$$

from (2.7) and (2.8). Applying (2.9) to (2.12) and comparing (2.11) finishes the proof of the proposition. □

3. Preliminary results on multiplicities of poles and some consequences of (2.2)

In this section we work only in even dimension d. The main points of this section are to define the multiplicities of poles of the resolvent and scattering matrix, and to prove Proposition 3.5 which identifies poles of the scattering matrix on Λ_{m+1} with zeros of a function defined on Λ_0.

3.1. Multiplicities of the poles of the resolvent.

This subsection recalls a result on the structure of the resolvent at a pole and defines two notions for the multiplicity of the pole of the resolvent.

A result which we shall need is the following lemma, which is essentially [9, Lemma 2.4] in a different setting. We do not give a proof, as it follows essentially identically the proof of that result. We use notation similar to [9, Lemma 2.4], but adapted for this context. In the statement below and later in this paper we use the notation λ^2 for the analytic continuation of the function which is equal to λ^2 for $\lambda \in \Lambda_0 \simeq \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

Lemma 3.1. *(cf.* [9, Lemma 2.4]*) If R has a pole at $\lambda_0 \in \Lambda$, then there is a finite $p > 0$ so that*

$$R(\lambda) = \sum_{k=1}^{p} \frac{A_k(\lambda_0)}{(\lambda^2 - \lambda_0^2)^k} + H_R(\lambda_0, \lambda)$$

where $H_R(\lambda_0, \lambda)$ is holomorphic near λ_0. There is a constant $0 < q < \infty$ so that

$$A_k(\lambda_0) = \sum_{l,m=1}^{q} a_k^{lm} \varphi_l \otimes \varphi_m, \tag{3.1}$$

with

$$\varphi_l \otimes \varphi_m(f) = \varphi_l \int f\varphi_m,$$

for $f \in \mathcal{H}$ *having* $f \mapsto f_{\restriction \mathbb{R}^d \setminus U}$ *with compact support. Moreover,* φ_m, $m = 1, ...q$ *satisfy*

$$\varphi_m \in \mathcal{H} \text{ if } \lambda_0 \in \partial\Lambda_0$$

$$\varphi_m \in \mathcal{D}_U \oplus e^{|x|(|\lambda_0| |\sin\arg\lambda_0|+\epsilon)} C_b^\infty(\mathbb{R}^d \setminus U) \text{ otherwise.}$$

If $a_k(\lambda_0)$ *denotes the matrix* $(a_k^{lm}(\lambda_0))_{1\leq l,m\leq q}$, *then* $a_1(\lambda_0)$ *is symmetric with rank* q, $d(\lambda_0) = a_1(\lambda_0)^{-1}a_2(\lambda_0)$ *is nilpotent, and* $a_k(\lambda_0) = a_1(\lambda_0)d(\lambda_0)^{k-1}$, $k > 1$.

We shall in fact need two notions related to the multiplicity of a pole of the resolvent. We first define the multiplicity μ_R of a pole of the resolvent R as follows. Given $\lambda_0 \in \Lambda$, define γ_{λ_0} to be a small circle centered at λ_0 that contains no poles of the resolvent except, possibly, a pole at λ_0. Define

$$\mu_R(\lambda_0) \stackrel{\text{def}}{=} \operatorname{rank} \int_{\gamma_{\lambda_0}} R(\lambda) 2\lambda d\lambda = \operatorname{rank} \int_{\gamma_{\lambda_0}} R(\lambda) d\lambda.$$

We note that by an argument just as in the proof of [9, Lemma 2.4], this is the same as the dimension of the image of the singular part of R at λ_0.

We need another, more restrictive, notion of multiplicity related to the resolvent of P satisfying the black box conditions. The need for this is related to the possibility of eigenvalues of $P_{\restriction \mathcal{H}_U}$; we provide an example below.

Let

$$\chi \in C_c^\infty(\mathbb{R}^d) \text{ satisfy } \chi \equiv 1 \text{ on } \overline{U} \tag{3.2}$$

where U is as in the black box assumptions on P of Section 2. Then define

$$\mu_{(1-\chi)R}(\lambda_0) \stackrel{\text{def}}{=} \operatorname{rank} \int_{\gamma_{\lambda_0}} (1-\chi) R(\lambda) d\lambda.$$

For any χ, $\tilde{\chi}$ both satisfying (3.2), unique continuation together with the expansion of Lemma 3.1 implies that $\mu_{(1-\chi)R}(\lambda_0) = \mu_{(1-\tilde{\chi})R}(\lambda_0)$.

It is clear that

$$\mu_{(1-\chi)R}(\lambda_0) \leq \mu_R(\lambda_0). \tag{3.3}$$

Moreover, the inequality can be strict, and it is the strictness of this inequality that makes having two definitions useful. Consider the following example. Let $\mathcal{O} \subset \mathbb{R}^d$ be an open bounded set with smooth boundary $\partial\mathcal{O}$. Suppose in addition that $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ has two connected components: $\mathbb{R}^d \setminus \overline{\mathcal{O}} = \Omega_{\text{ext}} \sqcup \Omega_{\text{bded}}$, where Ω_{ext} is unbounded and Ω_{bded} is bounded, and each is connected. An example of such an $\mathcal{O}$ is an annulus in $\mathbb{R}^2$. Then let $P = -\Delta$ with Dirichlet boundary conditions on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$. This operator P satisfies all the black box conditions. It is really the direct sum of two operators: one with discrete spectrum (the Dirichlet Laplacian on Ω_{bded}) and one with absolutely continuous spectrum (the Dirichlet Laplacian on $\Omega_{\rm ext}$). The inequality (3.3) is strict at points $e^{i\pi m}\lambda_1$, where $m \in \mathbb{Z}$ and λ_1^2 is an eigenvalue of P.

We include the following lemma now to further explain the relationship between the two notions of the multiplicity of a pole of the resolvent. The proof uses a result of [22], a representation of the scattering matrix, recalled here in Proposition 4.1. It also uses Lemma 4.4, the proof of which does not use the lemma below.

Lemma 3.2. *Suppose $\chi \in C_c^\infty(\mathbb{R}^d)$ satisfies (3.2). For $\lambda_0 \in \Lambda$,*

$$\mu_R(\lambda_0) = \mu_{(1-\chi)R}(\lambda_0) + \dim\{f \in \mathcal{H} : (P - \lambda_0^2)f = 0 \textit{ and } (1-\chi)f \equiv 0\}.$$

Proof. We use the notation of Lemma 3.1. By taking linear combinations of the φ_m if necessary and relabeling, we can assume that there is an $n \in \{1, 2, ..., q+1\}$ so that $(1-\chi)\varphi_m \equiv 0$ for $m = 1, 2, ..., n-1$ and so that $(1-\chi)\varphi_n, ..., (1-\chi)\varphi_q$ are linearly independent. This n is uniquely determined, and $n = 1 + \mu_R(\lambda_0) - \mu_{R(1-\chi)}(\lambda_0)$. If $n > 1$ then since $(P - \lambda_0^2)A_k(\lambda_0) = A_{k+1}(\lambda_0)$ and $A_{p+1}(\lambda_0) = 0$ (see [9, (2.22)]), we must have at least one eigenfunction of P in the span of $\{\varphi_1, ..., \varphi_{n-1}\}$, and must have $\lambda_0^2 \in \mathbb{R}$. Suppose there is an $l_0 \in \{1, 2, ..., n-1\}$ so that φ_{l_0} is *not* in the null space of $P - \lambda_0^2$. Since $(P - \lambda_0^2)\varphi_{l_0}$ is in the span of $\{\varphi_1, ..., \varphi_{n-1}\}$, it is in $\mathcal{H}$. But since $(P - \lambda_0^2)^p \varphi_m = 0$ for $m = 1, ..., q$, using that P is a self-adjoint and $\varphi_l \in \mathcal{H}$ this is a contradiction. Thus

$$\mu_R(\lambda_0) \leq \mu_{(1-\chi)R}(\lambda_0) + \dim\{f \in \mathcal{H} : (P - \lambda_0^2)f = 0 \text{ and } (1-\chi)f \equiv 0\}.$$

To see that

$$\mu_R(\lambda_0) \geq \mu_{(1-\chi)R}(\lambda_0) + \dim\{f \in \mathcal{H} : (P - \lambda_0^2)f = 0 \text{ and } (1-\chi)f \equiv 0\}$$

one can use the expression for $S(\lambda)$ from Proposition 4.1 along with Lemma 4.4. □

Thus we can see that it is the poles of $(1-\chi)R(\lambda)$ which are traditionally called resonances. Often the poles λ_0 with $\arg \lambda_0/\pi \in \mathbb{Z}$ are excluded, as they correspond to embedded eigenvalues.

3.2. **Multiplicities of the poles of the scattering matrix.** This subsection contains a number notions related to the multiplicity of poles of the scattering matrix. One such is measure of the multiplicities of the zeros and poles of a scalar function, which we shall denote m_{sc}, with the "sc" for scalar. Let f be a scalar function meromorphic on Λ, not identically 0, and let $\lambda_o \in \Lambda$. If $f(\lambda_0) = 0$, define $\mathrm{m}_{sc}(f, \lambda_0)$ to be the multiplicity of λ_0 as a zero of f. If f has a pole at λ_0, define $\mathrm{m}_{sc}(f, \lambda_0)$ to be minus the order of the pole of f at λ_0. If λ_0 is neither a pole nor a zero of f, set $\mathrm{m}_{sc}(f, \lambda_0) = 0$. Thus $\mathrm{m}_{sc}(f, \cdot)$ is positive at zeros and negative at poles.

Next we define the (maximum) multiplicity $\mu_{S,\max}(\lambda_0)$ of $\lambda_0 \in \Lambda$ as a pole of S. Near λ_0, we may for some finite p write

$$S(\lambda) = \sum_{j=1}^{p} \frac{B_j(\lambda_0)}{(\lambda - \lambda_0)^j} + H_S(\lambda_0, \lambda) \overset{\text{def}}{=} S_s(\lambda_0, \lambda) + H_S(\lambda_0, \lambda)$$

where $H_S(\lambda_0, \lambda)$ is holomorphic near λ_0. Note that $B_j(\lambda_0)$ is finite rank for $j = 1, ..., p$, and since the B_j are uniquely determined, so is $S_s(\lambda_0, \lambda)$. Then set

$$\mu_{S,\max}(\lambda_0) \overset{\text{def}}{=} -\,\mathrm{m}_{sc}(\det(I + S_s(\lambda_0, \lambda)), \lambda_0). \tag{3.4}$$

We discuss this definition further in Lemma 3.3. We note that our definition of the multiplicity of a pole of $S(\lambda)$ differs from one commonly used in scattering theory, that is

$$-\frac{1}{2\pi i}\,\mathrm{tr}\left(\int_{\gamma_{\lambda_0}} S^{-1}(\lambda) S'(\lambda) d\lambda\right) \tag{3.5}$$

where γ_{λ_0} is a small circle centered at λ_0 and enclosing no singularities of S or S^{-1} except possibly λ_0 (see, for example, [3, Equation 1.3]). Roughly speaking, the expression in (3.5) counts the multiplicity of the pole of S at λ_0 minus the multiplicity of the zero of S at λ_0; see Lemma 3.3. For many applications in scattering theory this is sufficient, as one expects all but a finite number of the poles in one half plane of $\mathbb{C}$, and all but a finite number of zeros in the other half plane of $\mathbb{C}$. The even dimensional Euclidean scattering case is more complicated. If $m \in \mathbb{Z}$ has $|m| > 1$, we expect in general that Λ_m contains both infinitely many poles and infinitely many zeros of S. Thus the definition (3.4) we use here counts the multiplicities of the poles without subtracting the multiplicities of the zeros.

The following lemma is well known, using that $S(\lambda) = (S^*(\overline{\lambda}))^{-1}$. We outline a proof, in part in an effort to make notions of multiplicities of a pole of the scattering matrix more transparent.

Lemma 3.3. *Let P be a self-adjoint operator satisfying the black box conditions recalled in Section 2. Let γ_{λ_0} be a small, positively oriented curve enclosing λ_0 and no zeros nor poles of $S(\lambda)$, except possibly at λ_0 itself. Then*

$$\frac{1}{2\pi i}\,\mathrm{tr}\int_{\gamma_{\lambda_0}} S'(\lambda) S^{-1}(\lambda) d\lambda = \mu_{S,\max}(\overline{\lambda}_0) - \mu_{S,\max}(\lambda_0) = \mathrm{m}_{sc}(\det S(\lambda), \lambda_0).$$

Proof. We note that $S(\lambda) - I$ is a compact operator which is finitely meromorphic on Λ; that is, the only singularities of $S(\lambda)$ are poles, and at each pole the singular part is of finite rank.[1] From [8, Theorem 3.1], one can write near $\lambda = \lambda_0$,

$$S(\lambda) = E(\lambda) D(\lambda) F(\lambda)$$

[1] Note that Λ does not include any points which project to the origin on the boundary of the physical half plane.

where $E,\ F$ are holomorphic with holomorphic inverses for λ in a neighborhood of λ_0. Moreover,

$$D(\lambda) = P_0 + \sum_{j=1}^{n} (\lambda - \lambda_0)^{k_j} P_j \tag{3.6}$$

and the $P_j,\ j = 0,\ 1, ...n$ are mutually orthogonal projections, $\operatorname{tr} P_j = 1$ for $j \geq 1$, $k_1, ..., k_n$ are integers, and $Q \stackrel{\text{def}}{=} I - \sum_{j=0}^{n} P_j$ is finite dimensional. In fact, using that $S(\lambda)$ has a meromorphic inverse, $Q = 0$. Moreover, the set $\{k_1, ..., k_n\}$ is uniquely determined by S and λ_0. Now

$$\mu_{S,\max}(\lambda_0) = \sum_{j=1}^{n} \max(0, -k_j)$$

where the k_j are as in (3.6). A comparison with [8] shows that this is what is called $P(S(\lambda_0))$ there. In the notation of [8]

$$N(S(\lambda_0)) = \sum_{j=1}^{n} \max(k_j, 0)$$

and in ours, using that $S(\lambda)^{-1} = S^*(\overline{\lambda})$,

$$\mu_{S,\max}(\overline{\lambda_0}) = \sum_{j=1}^{n} \max(k_j, 0).$$

Then from [8, Theorem 2.1] we have

$$\frac{1}{2\pi i} \operatorname{tr} \int_{\gamma_{\lambda_0}} S'(\lambda) S^{-1}(\lambda) d\lambda = \sum_{j=1}^{n} k_j = \mu_{S,\max}(\overline{\lambda_0}) - \mu_{S,\max}(\lambda_0).$$

Finally, the second equality of the lemma is a special case of [8, Theorem 5.1]. □

3.3. **A relation between poles of the scattering matrix on Λ_{m+1} and zeros of a scalar function on Λ_0.** Proposition 3.5 is proved in this section by fairly algebraic techniques.

The following lemma is a consequence of (2.2) and $S^*(\overline{\lambda}) = S^{-1}(\lambda)$.

Lemma 3.4. *Let d be even, $\lambda \in \Lambda$, and $m \in \mathbb{N}_0$. Then*

$$(S(\lambda)\mathcal{R}) \left(S(e^{i\pi}\lambda)\mathcal{R}\right) \cdots \left(S(e^{im\pi}\lambda)\mathcal{R}\right) = \left[(m+1)S(\lambda) - mI\right] \mathcal{R}^{m+1}.$$

Moreover,

$$\left(S(e^{im\pi}\lambda)\mathcal{R}\right) \left(S(e^{i(m-1)\pi}\lambda))\mathcal{R}\right) \cdots (S(\lambda)\mathcal{R}) = \mathcal{R}^m \left[(m+1)S(\lambda) - mI\right] \mathcal{R}.$$

Proof. The first identity trivially holds for $m = 0$. We now assume it holds for all integers between 0 and m inclusive, and show it holds for $m+1$. From (2.2),

$$\mathcal{R}S(e^{i(m+1)\pi}\lambda)\mathcal{R} = 2I - S^*(e^{-im\pi}\overline{\lambda}) = 2I - \left(S(e^{im\pi}\lambda)\right)^{-1}.$$

Multiplying both sides on the left by $S(e^{im\pi}\lambda)$ gives

$$S(e^{im\pi}\lambda)\mathcal{R}S(e^{i(m+1)\pi}\lambda)\mathcal{R} = 2S(e^{im\pi}\lambda) - I. \tag{3.7}$$

We note that if $m = 0$ this is the desired identity for $m+1 = 1$. So assume $m \geq 1$. Using the inductive hypothesis, multiply both sides of (3.7) on the left by

$$(S(\lambda)\mathcal{R}) \cdots \left(S(e^{i(m-1)\pi}\lambda)\mathcal{R}\right) = [mS(\lambda) - (m-1)I]\mathcal{R}^m$$

to obtain

$$\begin{aligned}
&(S(\lambda)\mathcal{R}) \left(S(e^{i\pi}\lambda)\mathcal{R}\right) \cdots \left(S(e^{i(m+1)\pi}\lambda)\mathcal{R}\right) \\
&= 2[S(\lambda)\mathcal{R} \cdots S(e^{i(m-1)\pi}\lambda)\mathcal{R}S(e^{im\pi}\lambda)\mathcal{R}]\mathcal{R} - [mS(\lambda) - (m-1)I]\mathcal{R}^m
\end{aligned}$$

Using the inductive hypothesis again, we find

$$\begin{aligned}
&(S(\lambda)\mathcal{R}) \left(S(e^{i\pi}\lambda)\mathcal{R}\right) \cdots \left(S(e^{i(m+1)\pi}\lambda)\mathcal{R}\right) \\
&= 2[(m+1)S(\lambda) - mI]\mathcal{R}^{1+m+1} - [mS(\lambda) - (m-1)I]\mathcal{R}^m \\
&= [(m+2)S(\lambda) - (m+1)I]\mathcal{R}^m
\end{aligned}$$

as desired.

The proof of the second equality is very similar. □

Proposition 3.5. *For $\lambda_0 \in \Lambda$, $m \in \mathbb{N}_0$,*

$$\begin{aligned}
\mathrm{m}_{sc}(\det((m+1)S(\lambda) - mI), \lambda_0) &= \sum_{j=0}^{m} \mathrm{m}_{sc}(\det S(\lambda), e^{ij\pi}\lambda_0) \\
&= \mu_{S,\max}(e^{i\pi(m+1)}\lambda_0) - \mu_{S,\max}(\lambda_0).
\end{aligned}$$

Before proving the proposition, we note that it shows that the poles of the scattering matrix on Λ_{m+1} correspond (with perhaps a finite number of exceptions) to the zeros of a scalar function $\det((m+1)S(\lambda) - mI)$ on Λ_0. This function is meromorphic on Λ_0, with at most finitely many poles (corresponding to eigenvalues of P) there. This is of course familiar in the odd-dimensional case, where it is well known, and has been extensively used, that with at most finitely many exceptions the poles of the scattering matrix in the nonphysical half plane correspond to zeros of the determinant of the scattering matrix in the physical half plane Λ_0.

We also note that using the symmetry of the poles of the scattering matrix which is implied by Proposition 2.1, poles of the scattering matrix on Λ_{-m}, $m \in \mathbb{N}$, can be identified with zeros of a scalar function using Proposition 3.5.

Proof. We give the proof for $m = 2l$ even; the proof for odd m is similar. Multiply both sides of the first identity of Lemma 3.4 (with m replaced by $2l$) by $\mathcal{R}$ on the right and rearrange slightly to get

$$S(\lambda)\left(\mathcal{R}S(e^{i\pi}\lambda)\mathcal{R}\right)S(e^{i2\pi}\lambda)\cdots\left(\mathcal{R}S(e^{i(2l-1)\pi}\lambda)\mathcal{R}\right)S(e^{i2l\pi}\lambda) = (2l+1)S(\lambda) - 2lI.$$

Since $S(\lambda)$ differs from the identity by a trace class operator, so do $\mathcal{R}S\mathcal{R}$ and $(2l+1)S - 2lI$. Thus we have

$$\det(S(\lambda))\det(\mathcal{R}S(e^{i\pi}\lambda)\mathcal{R})\cdots\det(\mathcal{R}S(e^{i(2l-1)\pi}\lambda)\mathcal{R})\det(S(e^{i2l\pi}\lambda)) = \det((2l+1)S(\lambda) - 2lI).$$

Using that $\mathcal{R}^2 = I$ and $\det(I + AB) = \det(I + BA)$ when A is trace class and B bounded,

$$\det(S(\lambda))\det(S(e^{i\pi}\lambda))\cdots\det(S(e^{i(2l-1)\pi}\lambda))\det(S(e^{i2l\pi}\lambda)) = \det((2l+1)S(\lambda) - 2lI).$$

This gives us

$$\sum_{j=0}^{2l} \mathrm{m}_{sc}(\det(S(\lambda)), e^{i\pi j}\lambda_0) = \mathrm{m}_{sc}(\det\left((2l+1)S(\lambda) - 2lI\right), \lambda_0). \tag{3.8}$$

By (2.2), $\lambda_0 \in \Lambda$ is a pole of $S(\lambda)$ if and only if $e^{\pi i}\overline{\lambda_0}$ is a pole of $S^*(\lambda)$, and the (maximum) multiplicities coincide. Applying this and Lemma 3.3,

$$\begin{aligned}\sum_{j=0}^{2l} \mathrm{m}_{sc}(\det(S(\lambda)), e^{i\pi j}\lambda_0) &= \sum_{j=0}^{2l}\left(-\mu_{S,\max}(e^{i\pi j}\lambda_0) + \mu_{S,\max}(e^{-i\pi j}\overline{\lambda}_0)\right) \\ &\sum_{j=0}^{2l}\left(-\mu_{S,\max}(e^{i\pi j}\lambda_0) + \mu_{S,\max}(e^{i\pi(j+1)}\lambda_0)\right) \\ &= \mu_{S,\max}(e^{i\pi(2l+1)}\lambda_0) - \mu_{S,\max}(\lambda_0).\end{aligned}$$

Combined with (3.8), this completes the proof. □

4. Poles of the resolvent and poles of the scattering matrix

In this section we work only in even dimensions d. The main result of this section is Theorem 4.5, an identification between the poles of the resolvent and the poles of the scattering matrix. While analogs of this result are well known both in odd dimensions and for points in Λ_1 and Λ_{-1} (see e.g. [11, 18, 22, 26]), we are unaware of a proof in the literature which includes the other sheets of Λ.

We shall use [22, Proposition 2.1] which we recall here for the convenience of the reader. We have changed the notation to be consistent with the notation of this paper. We remark that there are a number of similar representations of the scattering matrix in the literature; see, for example, [21, Section 2] or [35, Section 3]. We recall that our Hilbert space $\mathcal{H}$ has an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_U \oplus L^2(\mathbb{R}^d \setminus U)$$

where $U \subset \mathbb{R}^d$ is a bounded open set.

Proposition 4.1. *(*[22, Proposition 2.1]*) For $\phi \in C_c^\infty(\mathbb{R}^d)$, let us denote by*

$$\mathbb{E}_\pm^\phi(\lambda) : L^2(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$$

the operator with the kernel $\phi(x)\exp(\pm i\lambda\langle x, \omega\rangle)$. Let us choose $\chi_i \in C_c^\infty(\mathbb{R}^d)$, $i = 1,\ 2,\ 3$, such that $\chi_i \equiv 1$ near U and $\chi_{i+1} \equiv 1$ on $\operatorname{supp}\chi_i$.

Then for $0 < \arg\lambda < \pi$ we have $S(\lambda) = I + A(\lambda)$, where

$$A(\lambda) = i\pi(2\pi)^{-d}\lambda^{(d-1)/2}\mathbb{E}_+^{\chi_3}(\lambda)[\Delta, \chi_1]R(\lambda)[\Delta, \chi_2]^t\mathbb{E}_-^{\chi_3}(\lambda)$$

where ${}^t\mathbb{E}$ denotes the transpose of $\mathbb{E}$. The identity holds for $\lambda \in \Lambda$ by analytic continuation.

For $\lambda > 0$ let $\Phi(\lambda, x, \omega)$ be the function satisfying

$$(P - \lambda^2)\Phi(\lambda, x, \omega) = 0$$

$$\Phi(\lambda, r\theta, \omega) = e^{-i\lambda r\theta\cdot\omega} + r^{-(d-1)/2}e^{i\lambda r}\left(s_+(\theta, \lambda, -\omega) + o(r)\right) \text{ as } r \to \infty.$$

Here we understand that P acts in the x variable, and $r > 0$, $\theta \in \mathbb{S}^{d-1}$.The function Φ has a meromorphic extension to $\lambda \in \Lambda$ which we denote in the same way. Note that if $\chi_1 \in C_c^\infty(\mathbb{R}^d)$ satisfies the conditions of Proposition 4.1, then

$$\Phi(\lambda, x, \omega) = (1 - \chi_1)e^{-i\lambda x\cdot\omega} - R(\lambda)[\Delta, \chi_1]e^{-i\lambda x\cdot\omega}.$$

We shall also denote by $\Phi(\lambda)$ the operator from $L^2(\mathbb{S}^{d-1})$ to $\mathcal{H}_U \oplus L^2_{\text{loc}}(\mathbb{R}^d \setminus U)$ which is given by

$$(\Phi(\lambda)f)(x) = \int_{\mathbb{S}^{d-1}} f(\omega)\Phi(\lambda, x, \omega)dS_\omega,$$

and by $\Phi^t(\lambda)$ the transpose. By Stone's formula, for $\lambda > 0$

$$R(\lambda) - R(\lambda e^{i\pi}) = \alpha_d\lambda^{d-2}\Phi(\lambda)\Phi^t(\lambda e^{i\pi}), \tag{4.1}$$

where $\alpha_d = -i(2\pi)^{1-d}/2$; compare [16, (2.26)]. Since both sides have meromorphic extensions to Λ, the equality holds for $\lambda \in \Lambda$ away from the poles.

The next two lemmas pave the way for Lemma 4.4, which expresses the resolvent at $e^{im\pi}\lambda$ in terms of $S(e^{im\pi}\lambda)$, $R(\lambda)$, and $\Phi(\lambda)$.

Lemma 4.2. *For $\lambda \in \Lambda$,*

$$\Phi(\lambda e^{i\pi}) = \Phi(\lambda)\mathcal{R}S^*(\overline{\lambda}).$$

Proof. Although this well known, we sketch the proof. For $\lambda > 0$, and $x \in \mathbb{R}^n \setminus U$,

$$\Phi(\lambda e^{i\pi}, x, \omega) - \left(\Phi(\lambda)\mathcal{R}S^*(\overline{\lambda})\right)(x,\omega)$$
$$= e^{i\lambda|x|}|x|^{-(d-1)/2}(\beta(x/|x|,\omega)) + O(|x|^{-(d+1)/2}) \text{ as } |x| \to \infty$$

for some function $\beta \in C^\infty(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$. By Rellich's uniqueness theorem, since $\Phi(\lambda e^{i\pi}) - \Phi(\lambda)\mathcal{R}S^*(\overline{\lambda})$ is in the null space of $P - \lambda^2$, this is enough to show the difference is 0. The general result follows by analytic continuation. □

Lemma 4.3. *For $m \in \mathbb{N}$ and $\lambda \in \Lambda$,*

$$\Phi(\lambda e^{im\pi})\Phi^t(\lambda e^{i(m+1)\pi})$$
$$= \Phi(\lambda)\left[(m+1)^2\mathcal{R}^{m+1}S^*(\overline{\lambda}e^{-im\pi})\mathcal{R}^m - m^2\mathcal{R}^m S^*(\overline{\lambda}e^{-i(m-1)\pi})\mathcal{R}^{m-1} - 2m\mathcal{R}\right]\Phi^t(\lambda).$$

Proof. By repeatedly applying Lemma 4.2 and the identity $(\mathcal{R}S^*)^t = \mathcal{R}S^*$, we have

$$\Phi(\lambda e^{im\pi})\Phi^t(\lambda e^{i(m+1)\pi}) = \tag{4.2}$$
$$\Phi(\lambda)\mathcal{R}S^*(\overline{\lambda})\mathcal{R}S^*(\overline{\lambda}e^{-i\pi})\cdots\mathcal{R}S^*(\overline{\lambda}e^{-i(m-1)\pi})\mathcal{R}S^*(\overline{\lambda}e^{-im\pi})\cdot\mathcal{R}S^*(\overline{\lambda}e^{-i(m-1)\pi})\cdots\mathcal{R}S^*(\overline{\lambda})\Phi^t(\lambda).$$

Lemma 3.4 implies that for $p \in \mathbb{N}_0$

$$\mathcal{R}S^*(e^{ip\pi}\lambda)\mathcal{R}S^*(\lambda e^{i(p-1)\pi})\cdots\mathcal{R}S^*(\lambda) = \mathcal{R}^{p+1}\left[(p+1)S^*(\lambda) - pI\right].$$

Applying this identity with $\lambda e^{ip\pi}$ replaced by $\overline{\lambda}$ and with $p = m$, we find that (4.2) is

$$\Phi(\lambda)\mathcal{R}^{m+1}\left[(m+1)S^*(\overline{\lambda}e^{-im\pi}) - mI\right]\mathcal{R}S^*(\overline{\lambda}e^{-i(m-1)\pi})\cdots\mathcal{R}S^*(\overline{\lambda})\Phi^t(k).$$

Distributing and then using the second part of Lemma 3.4 twice, this is

$$\Phi(\lambda)\mathcal{R}^{m+1}(m+1)\left[(m+1)S^*(\overline{\lambda}e^{-im\pi}) - mI\right]\mathcal{R}^m\Phi^t(\lambda)$$
$$- m\Phi(\lambda)\mathcal{R}^m\left[mS^*(\overline{\lambda}e^{-i(m-1)\pi}) - (m-1)I\right]\mathcal{R}^{m-1}\Phi^t(\lambda)$$
$$= \Phi(\lambda)\left[(m+1)^2\mathcal{R}^{m+1}S^*(\overline{\lambda}e^{-im\pi})\mathcal{R}^m - m^2\mathcal{R}^m S^*(\overline{\lambda}e^{-i(m-1)\pi})\mathcal{R}^{m-1} - 2m\mathcal{R}\right]\Phi^t(\lambda).$$

□

The next lemma allows us to express the resolvent on Λ_m, $m \in \mathbb{N}$, in terms of the resolvent on Λ_0, the generalized eigenfunctions $\Phi(\lambda)$ on Λ_0, and the scattering matrix S on Λ_m.

Lemma 4.4. *Let P satisfy the general black box conditions recalled in Section 2. Then for $m \in \mathbb{N}$,*

$$R(e^{im\pi}\lambda) - R(\lambda) = \alpha_d m\lambda^{d-2}\Phi(\lambda)\mathcal{R}^{m+1}\left[mS(e^{im\pi}\lambda) - (m+1)I\right]\mathcal{R}^m\Phi^t(\lambda) \tag{4.3}$$

Proof. We have

$$
\begin{aligned}
R(e^{im\pi}\lambda)-R(\lambda) &= \sum_{j=1}^{m}(R(e^{ij\pi}\lambda)-R(e^{i(j-1)\pi}\lambda)) \\
&= -\sum_{j=1}^{m}\alpha_d\lambda^{d-2}\Phi(e^{i(j-1)\pi}\lambda)\Phi^t(\lambda e^{ij\pi}) \\
&= -\sum_{j=1}^{m}\alpha_d\lambda^{d-2}\Phi(\lambda)\left[j^2\mathcal{R}^jS^*(\overline{\lambda}e^{-i(j-1)\pi})\mathcal{R}^{j-1}\right. \\
&\qquad \left. -(j-1)^2\mathcal{R}^{j-1}S^*(\overline{\lambda}e^{-i(j-2)\pi})\mathcal{R}^{j-2}-2(j-1)\mathcal{R}\right]\Phi^t(\lambda)
\end{aligned}
$$

where the second equality follows from (4.1) and the third follows from Lemma 4.3.

Now

$$
\begin{aligned}
&\sum_{j=1}^{m}\left[j^2\mathcal{R}^jS^*(\overline{\lambda}e^{-i(j-1)\pi})\mathcal{R}^{j-1}-(j-1)^2\mathcal{R}^{j-1}S^*(\overline{\lambda}e^{-i(j-2)\pi})\mathcal{R}^{j-2}-2(j-1)\mathcal{R}\right] \\
&\qquad\qquad = m^2\mathcal{R}^mS^*(\overline{\lambda}e^{-i(m-1)\pi})\mathcal{R}^{m-1}-m(m-1)\mathcal{R}
\end{aligned}
$$

using the fact that the first two summands telescope. Since

$$S^*(\overline{\lambda}e^{-i(m-1)\pi}) = 2I-\mathcal{R}S(e^{im\pi}\lambda)\mathcal{R}$$

from (2.2), this proves the lemma. □

We now turn more directly to the central result of this section.

Theorem 4.5. *Let d be even and P satisfy the general black box conditions recalled in Section 2, and let $\chi\in C_c^\infty(\mathbb{R}^d)$ have $\chi\equiv 1$ on $\overline{U}$. Then for $\lambda_0\in\Lambda$,*

$$\mu_{(1-\chi)R}(\lambda_0)-\mu_{(1-\chi)R}(\overline{\lambda_0}) = -\,\mathrm{m}_{sc}(\det S(\lambda),\lambda_0) = \mu_{S,\max}(\lambda_0)-\mu_{S,\max}(\overline{\lambda_0})$$

and

$$\mu_R(\lambda_0)-\mu_R(\overline{\lambda_0}) = -\,\mathrm{m}_{sc}(\det S(\lambda),\lambda_0) = \mu_{S,\max}(\lambda_0)-\mu_{S,\max}(\overline{\lambda_0}).$$

We note the second equality in each displayed equation follows from Lemma 3.3.

This result is well-known in odd dimensions, and in even dimensions is known for $\lambda\in\Lambda_1\cup\Lambda_{-1}$, [11, 18, 22, 26]. As we are unaware of a proof in the literature valid for other points in Λ for even dimensions d, we include it in this section. The proof we shall give of Theorem 4.5 follows rather closely the proof of an analogous result of Borthwick and Perry for asymptotically hyperbolic manifolds given in [3], and a similar result (for a subset of Λ) in [22]. The paper [3] uses Agmon's perturbation theory of resonances [1] together with some ideas of Klopp and Zworski's paper [12] to prove a result on generic simplicity of resonances (away from certain points). The analog of Theorem 4.5 is then proved first for situations in which the resonances are simple, and then for the general case using as an ingredient the genericity result.

Denote by Y_l^m, $l = 0, 1, 2, ..., m = 1, 2, ...m(l)$ a complete orthonormal set of the spherical harmonics on $\mathbb{S}^{d-1}$, where $m(l) = \frac{2l+d-2}{d-2}\binom{l+d-3}{d-3}$. These eigenfunctions of the Laplacian $\Delta_{\mathbb{S}^{d-1}}$ on $\mathbb{S}^{d-1}$ satisfy

$$-\Delta_{\mathbb{S}^{d-1}} Y_l^m = l(l+d-2)Y_l^m,\ l = 0,\ 1,\ 2, ...\ m = 1,\ 2,\ ...,\ m(l).$$

From [29, Lemma 3] [2]

$$e^{i\lambda x\cdot\omega} = (2\pi)^{d/2}\sum_{l=0}^{\infty}\sum_{m=1}^{m(l)} i^l \overline{Y}_l^m(\theta) Y_l^m(\omega)(\lambda r)^{1-d/2} J_{l+d/2-1}(\lambda r),\ x = r\theta. \tag{4.4}$$

This equality is classical for $d = 3$.

The next two lemmas prove special cases of Theorem 4.5 under an assumption of simplicity of the pole of $(1-\chi)R$ at λ_0.

Lemma 4.6. *Let P be self-adjoint and satisfy the other general black box conditions recalled in Section 2 and let $\chi \in C_c^\infty(\mathbb{R}^d)$, with $\chi \equiv 1$ on $\overline{U}$. Let $\lambda_0 \in \Lambda$ and suppose $\mu_{(1-\chi)R}(\lambda_0) \leq 1$. Then $\mu_{(1-\chi)R}(\lambda_0) = \mu_{S,\max}(\lambda_0)$.*

Proof. When $\mu_{(1-\chi)R}(\lambda_0) \leq 1$, from Proposition 4.1 we see that we must have $\mu_{S,\max}(\lambda_0) \leq \mu_{(1-\chi)R}(\lambda_0)$.

So suppose $\mu_{(1-\chi)R}(\lambda_0) = 1$. It follows from Lemma 3.1, Lemma 3.2 and its proof that R has a simple pole at λ_0. It is enough to show that $\mu_{S,\max}(\lambda_0) \geq 1$. From the proof of [27, Lemma 3.2], [22, Equation 4.2] or [32, Equation 4.1],

$$(1-\chi)R(\lambda) = (1-\chi)R_0(\lambda)(I + K(\lambda))^{-1} \tag{4.5}$$

where $R_0(\lambda)$ is the resolvent for $-\Delta$ on $\mathbb{R}^d$ and $K(\lambda) : \mathcal{H}_U \oplus L_0^2(\mathbb{R}^d \setminus U) \to \mathcal{H}_U \oplus L_0^2(\mathbb{R}^d \setminus U)$ is a compact operator. Thus since $(1-\chi)R(\lambda)$ has a simple pole of rank 1 at λ_0 by assumption, the residue of $(1-\chi)R$ at λ_0 is of the form (see Lemma 3.1)

$$a(1-\chi)\varphi \otimes \varphi$$

where a is a nonzero constant, $(P - \lambda_0^2)\varphi = 0$, $\varphi \not\equiv 0$, and $(1-\chi)\varphi \not\equiv 0$. Moreover, since by (4.5) and the more explicit expressions for $K(\lambda)$ found in the references given, $(1-\chi)\varphi = (1-\chi)R_0(\lambda_0)g$ for some $g \in \mathcal{H}_U \oplus L_0^2(\mathbb{R}^d \setminus U)$, we have

$$(1-\chi)\varphi(r\theta) = \sum_{l,m} c_{lm} Y_l^m(\theta) r^{1-d/2} H^{(1)}_{l+d/2-1}(r\lambda_0),\ \text{for sufficiently large } r. \tag{4.6}$$

By unique continuation not all of the c_{lm} can be 0.

To show that $\mu_{S,\max}(\lambda_0) \geq 1$, by Proposition 4.1 it is enough to show that

$$C^\infty(\mathbb{S}^{d-1}) \ni \int \varphi(x)[\Delta, \chi_j] e^{i\lambda x\cdot\omega} dx \not\equiv 0,\ j = 1, 2$$

[2] The proof in [29] holds for $d \geq 3$ at least; the proof for $d = 2$ follows from the Jacobi-Anger expansion.

for χ_j as in the statement of the proposition. Using Green's theorem,

$$\begin{aligned}\int \varphi(x)[\Delta,\chi_j]e^{i\lambda x\cdot\omega}dx &= \int \varphi(x)[\Delta,\chi_j-1]e^{i\lambda x\cdot\omega}dx \\ &= -\int_{|x|=R}\left(\varphi(x)\frac{\partial}{\partial|x|}e^{i\lambda x\cdot\omega} - e^{i\lambda x\cdot\omega}\frac{\partial}{\partial|x|}\varphi\right)dS \end{aligned} \tag{4.7}$$

if $R > R_0$, where R_0 satisfies $\operatorname{supp}\chi_j \subset B(0,R_0) = \{x\in\mathbb{R}^d : |x| < R_0\}$. Notice the right hand side of (4.7) is independent R satisfying this condition. Applying this with the expansions (4.6) and (4.4) gives, for R sufficiently large

$$\int \varphi(x)[\Delta,\chi_j]e^{i\lambda_0 x\cdot\omega}dx = \sum Y_l^m(\omega)c_{lm}(2\pi)^{d/2}i^l\lambda_0^{1-d/2}g_{lm} \tag{4.8}$$

where

$$\begin{aligned} g_{lm} = R^{d/2}\Bigg[J_{l+d/2-1}(\lambda_0 R)\frac{\partial}{\partial R}\left(R^{1-d/2}H^{(1)}_{l+d/2-1}(\lambda_0 R)\right) \\ -H^{(1)}_{l+d/2-1}(\lambda_0 R)\frac{\partial}{\partial R}\left(R^{1-d/2}J_{l+d/2-1}(\lambda_0 R)\right)\Bigg].\end{aligned}$$

Of course, g_{lm} is in fact independent of R.

Now we use that if $\alpha\in\mathbb{N}$, $J_\alpha(\lambda)$ and $H_\alpha(\lambda)$ are holomorphic functions on Λ satisfying, for $m\in\mathbb{Z}$,

$$J_\alpha(e^{im\pi}\lambda) = e^{im\alpha\pi}J_\alpha(\lambda)$$

and

$$H^{(1)}_\alpha(e^{im\pi}\lambda) = (-1)^{m\alpha}\left[H^{(1)}_\alpha(\lambda) - 2mJ_\alpha(\lambda)\right]$$

[19, pg. 239, 4.13]. Using that $\lambda_0 = e^{im\pi}\lambda_1$ for some $m\in\mathbb{Z}$ and λ_1 with $0\le\arg\lambda_1 < \arg\pi$,

$$g_{lm} = R\left[J_{l+d/2-1}(\lambda_1 R)\frac{\partial}{\partial R}H^{(1)}_{l+d/2-1}(\lambda_1 R) - H^{(1)}_{l+d/2-1}(\lambda_1 R)\frac{\partial}{\partial R}J_{l+d/2-1}(\lambda_1 R)\right]. \tag{4.9}$$

We recall the expansions, valid as $|z|\to\infty$,

$$H^{(1)}_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2}e^{i(z-\pi\nu/2-\pi/4)}\left(1+O(|z|^{-1})\right),\ \text{if } -\pi+\delta\le\arg z\le 2\pi-\delta \tag{4.10}$$

and

$$J_\nu(z) = \left(\frac{1}{2\pi z}\right)^{1/2}e^{-i(z-\nu\pi/-\pi/4)}\left(1+O(1/|z|)\right)\ \text{if } \delta<\arg z<\pi-\delta \tag{4.11}$$

[20, Equation 9.2.7] and [20, Equation 9.2.1]. In each case, $\delta>0$ and the principal branch of the square root is taken. If we apply these expansions and the related one for the derivatives (See [20, Equations 9.2.11 and 9.2.13]; the leading order terms can be obtained by differentiating the leading order terms of (4.10) and (4.11).) we find

$$g_{lm} = 2i/\pi + O(1/R) \text{ as } R \to \infty$$

provided $\arg \lambda_1 \neq 0$. Since g_{lm} is in fact constant, we have shown that $g_{lm} = 2i/\pi$. Thus by (4.8), $\int \varphi(x)[\Delta, \chi_j] e^{i\lambda_0 x\cdot\omega} dx \not\equiv 0$ and S has a simple pole with a residue of rank 1 at λ_0 if $\arg \lambda_1 \neq 0$.

We finish our proof by noting that our assumption that λ_0 is a simple pole of $(1-\chi)R(\lambda)$ means that $\arg(\lambda_0)/\pi \notin \mathbb{Z}$. To see this, recall that under our assumption that P is self-adjoint this is well known for $\arg \lambda_0 = 0$. Then Proposition 3.5, combined with the fact that S is unitary on $\arg \lambda = 0$ since P is selfadjoint, shows that $S(\lambda)$ cannot have poles with $(\arg \lambda)/\pi \in \mathbb{Z}$. Finally, (4.3) and its adjoint equation show that $(1-\chi)R(\lambda)$ cannot have poles with $(\arg \lambda)/\pi \in \mathbb{Z}$. □

The next lemma builds a bit on the previous one.

Lemma 4.7. *Let P satisfy the general black box conditions recalled in Section 2, and let $\chi \in C_c^\infty(\mathbb{R}^d)$, with $\chi \equiv 1$ on $\overline{U}$. Let $\lambda_0 \in \Lambda$ and suppose both $\mu_{(1-\chi)R}(\lambda_0) \leq 1$ and $\mu_{(1-\chi)R}(\overline{\lambda_0}) \leq 1$. Then*

$$\mu_R(\lambda_0) - \mu_R(\overline{\lambda_0}) = -\, \mathrm{m}_{sc}(\det S(\lambda), \lambda_0).$$

Proof. Applying Lemmas 3.3 and 4.6 we see that

$$\mu_{(1-\chi)R}(\lambda_0) - \mu_{(1-\chi)R}(\overline{\lambda_0}) = -m_{sc}(\det S(\lambda), \lambda_0). \tag{4.12}$$

It follows from Lemmas 3.2 and 4.4 that for $m \in \mathbb{Z}$

$$\mu_R(\lambda e^{im\pi}) - \mu_{(1-\chi)R}(\lambda e^{im\pi}) = \mu_R(\lambda) - \mu_{(1-\chi)R}(\lambda). \tag{4.13}$$

Next we note that if $\mu_R(\lambda) - \mu_{(1-\chi)R}(\lambda) > 0$ then $2(\arg \lambda)/\pi \in \mathbb{Z}$ by Lemma 3.2 and using the selfadjointness of P. But if this holds, then $(\arg \lambda - \arg \overline{\lambda})/\pi \in \mathbb{Z}$. Thus applying (4.12) and (4.13) finishes the proof. □

The following proposition extends the main result of [12] and is very closely related to Theorem 5.1 of [3]. Below we use the notation $R_{P+V}(\lambda)$ for the meromorphic continuation of the resolvent of $P+V$ to Λ.

Proposition 4.8. *Suppose P satisfies the black box conditions recalled in Section 2, and let $U \subset \mathbb{R}^d$ be the bounded open set as in the statement of the hypotheses on P, with $Pf = -\Delta f$ for $f \in H^2(\mathbb{R}^d \setminus U)$. Let $\chi \in C_c^\infty(\mathbb{R}^d)$ satisfy $\chi \equiv 1$ on $\overline{U}$. Then the set of potentials $V \in C_c^\infty(\mathbb{R}^d \setminus \overline{U}; \mathbb{R})$ for which all poles of $(1-\chi)R_{P+V}$ are simple with rank 1 residues is a dense G_δ set in $C_c^\infty(\mathbb{R}^d \setminus \overline{U}; \mathbb{R})$.*

In fact, [12] proved that generically the resonances in $\Lambda_1 \cup \Lambda_{-1}$ are simple. Here we say a statement holds generically if it holds for a dense G_δ set. The proof of [12] uses complex scaling and so does not obviously immediately extend to other sheets of Λ. The paper [3] proved the generic simplicity of resonances for $\Delta_g + V$ on asymptotically hyperbolic manifolds. There the main ingredient of the proof is Agmon's perturbation theory for resonances [1], which is sufficiently general to be applicable to the situation described in Proposition 4.8. A second part of the proof is a unique continuation theorem, also valid here. The proof of Proposition 4.8 in this setting follows so closely the proof of [3, Theorem 5.1] that we do not repeat it here.

Proof of Theorem 4.5. Let γ_{λ_0} be a small circle centered at λ_0 which does not enclose any other poles of $R_P(\lambda)$ and so that γ_{λ_0} encloses no poles of $R_P^*(\overline{\lambda})$ except possibly one at λ_0. Moreover, we require that both $R_P(\lambda)$ and $R_P^*(\overline{\lambda})$ are holomorphic on γ_{λ_0} itself.

For $t \in \mathbb{R}$ and $V \in C_c^\infty(\mathbb{R}^d \setminus \overline{U})$, we shall denote by R_{P+tV} the resolvent of $P + tV$ and similarly by S_{P+tV} the associated scattering matrix. For any $V \in C_c^\infty(\mathbb{R}^d \setminus \overline{U})$ so that no poles of $R_{P+tV}(\lambda)$ cross γ_{λ_0} for $t \in [0, 1]$, the operator-valued integral

$$\int_{\gamma_{\lambda_0}} R_{P+tV}(\lambda) d\lambda$$

is continuous for $t \in [0, 1]$. Hence for such V and for all $t \in [0, 1]$, the rank of the residue is constant and equal to its value at $t = 0$:

$$\operatorname{rank} \int_{\gamma_{\lambda_0}} R_{P+tV}(\lambda) d\lambda = \mu_{R_P}(\lambda_0) \tag{4.14}$$

and

$$\operatorname{rank} \int_{\gamma_{\lambda_0}} (1 - \chi) R_{P+tV}(\lambda) d\lambda = \mu_{(1-\chi)R_P}(\lambda_0). \tag{4.15}$$

We note that it follows from Lemma 3.1 that

$$\operatorname{rank} \int_{\gamma_{\lambda_0}} R_{P+tV}(\lambda)(1 - \chi) = \operatorname{rank} \int_{\gamma_{\lambda_0}} (1 - \chi) R_{P+tV}(\lambda).$$

Likewise, if no poles of $R_{P+tV}^*(\overline{\lambda})$ cross γ_{λ_0} for $t \in [0, 1]$, then for all $t \in [0, 1]$

$$\operatorname{rank} \int_{\gamma_{\lambda_0}} R_{P+tV}^*(\overline{\lambda}) d\lambda = \mu_{R_P}(\overline{\lambda_0}) \tag{4.16}$$

and

$$\operatorname{rank} \int_{\gamma_{\lambda_0}} R_{P+tV}^*(\overline{\lambda})(1 - \chi) d\lambda = \mu_{(1-\chi)R_P}(\overline{\lambda_0}), \tag{4.17}$$

using that the rank of A^* is equal to the rank of A.

By Proposition 4.8, we may choose $V \in C_c^\infty(\mathbb{R}^d \setminus \overline{U}; \mathbb{R})$ so that no poles of either $R_{P+tV}(\lambda)$ or $R_{P+tV}^*(\overline{\lambda})$ cross γ_{λ_0} for $t \in [0, 1]$ and so that $(1 - \chi) R_{P+tV}(\lambda)$ has simple poles with rank one residues for some $t_0 \in [0, 1]$.

Then we use (4.14)- (4.17), (4.12), and Lemma 4.7 to show that
(4.18)
$$\mu_{R_P}(\lambda_0) - \mu_{R_P}(\overline{\lambda_0}) = \mu_{(1-\chi)R_P}(\lambda_0) - \mu_{(1-\chi)R_P}(\overline{\lambda_0}) = -\,\mathrm{m}_{sc}(\det(S_{P+t_0V}(\lambda)), \lambda_0).$$
When V is chosen as above, by Lemma 3.3 and using the choice of γ_{λ_0},
$$\mathrm{m}_{sc}(\det(S_P(\lambda)), \lambda_0) = \frac{1}{2\pi i}\,\mathrm{tr} \int_{\gamma_{\lambda_0}} S_P^{-1}(\lambda) S_P'(\lambda) d\lambda.$$
Since neither zeros nor poles of $S_P(\lambda)$ lie on γ_{λ_0} for $t \in [0,1]$, the operator-valued integral
$$\frac{1}{2\pi i}\,\mathrm{tr} \int_{\gamma_{\lambda_0}} S_{P+tV}^{-1}(\lambda) S_{P+tV}'(\lambda) d\lambda$$
is a continuous function of $t \in [0,1]$, and hence, being an integer, is the constant $\mathrm{m}_{sc}(\det(S_P(\lambda)), \lambda_0)$. Combining this observation with (4.18) proves the theorem. □

We give a corollary to Theorem 4.5 which may be helpful in studying resonances on Λ_m.

Corollary 4.9. *Under the hypotheses of Theorem 4.5, for $\lambda_1 \in \Lambda$, $m \in \mathbb{N}$,*
$$\begin{aligned}\mu_{(1-\chi)R}(\lambda_1 e^{im\pi}) - \mu_{(1-\chi)R}(\lambda_1) &= -\,\mathrm{m}_{sc}(\det(mS(\lambda) - (m-1)I), \lambda_1)\\ &= \mu_R(\lambda_1 e^{im\pi}) - \mu_R(\lambda_1).\end{aligned}$$

Proof. We note first that $R(\lambda) = R^*(e^{i\pi}\overline{\lambda})$ means that $\mu_R(\lambda) = \mu_R(e^{i\pi}\overline{\lambda})$, and similarly for $\mu_{(1-\chi)R}$. Using this and applying Theorem 4.5 gives $\mu_R(\lambda_0) - \mu_R(e^{i\pi}\lambda_0) = -\,\mathrm{m}_{sc}(\det S, \lambda_0)$ any $\lambda_0 \in \Lambda$. Repeatedly using this identity with λ_0 replaced by λ_1, $e^{i\pi}\lambda_1$,..., $e^{i(m-1)\pi}\lambda_1$ in turn and adding gives

(4.19)
$$\begin{aligned}\mu_R(\lambda_1) - \mu_R(e^{i\pi}\lambda_1) + \mu_R(e^{i\pi}\lambda_1) - \mu_R(e^{i2\pi}\lambda_1) + \cdots + \mu_R(e^{i(m-1)\pi}\lambda_1) - \mu_R(e^{im\pi}\lambda_1)\\ = -\,\mathrm{m}_{sc}(\det S, \lambda_1) - \mathrm{m}_{sc}(\det S, e^{i\pi}\lambda_1) - \cdots - \mathrm{m}_{sc}(\det S, e^{i(m-1)\pi}\lambda_1).\end{aligned}$$
Applying Proposition 3.5, we find
$$\mu_R(\lambda_1) - \mu_R(e^{im\pi}\lambda_1) = -\,\mathrm{m}_{sc}\left(\det(mS(\lambda) - (m-1)I), \lambda_1\right).$$
The result for $\mu_{R(1-\chi)}$ follows similarly. □

5. Purely imaginary poles and (2.2)

The purpose of this section is to prove some consequences of (2.2) regarding poles of the scattering matrix in even dimensions. Among other things, we shall point out the importance of the distinction between (2.2) and (2.1) when related to the question of the existence of purely imaginary poles of the scattering matrix or resolvent. In this section we again assume d is even.

Theorem 5.1. *Let $\sigma > 0$, and denote by $i\sigma$ the point of Λ with argument $\pi/2$ and norm σ. If $S(i\sigma) - I$ has only purely imaginary eigenvalues, then $S(\lambda)$ is analytic in a neighborhood of $e^{i(m\pi+\pi/2)}\sigma$, $m \in \mathbb{Z} \setminus \{0\}$. Moreover, if $-\sigma^2$ is not an eigenvalue of P, then P does not have a resolvent resonance at $e^{i(m\pi+\pi/2)}\sigma$.*

Proof. We note that if $S(i\sigma) - I$, a compact operator, has only purely imaginary eigenvalues, then $(m+1)S(i\sigma) - mI = (m+1)(S(i\sigma) - I) + I$ has no nontrivial null space. This gives $\mathrm{m}_{sc}(\det((m+1)S(\lambda) - mI), i\sigma) \leq 0$. Thus if $m > 1$ by applying Proposition 3.5 we see that S cannot have a pole at $e^{i(m\pi+\pi/2)}\sigma$.

Again assuming $m > 0$, we note that S has a pole at $e^{i(-\pi m+\pi/2)}\sigma$ if and only if S has a pole at $e^{i\pi}e^{i(\pi m-\pi/2)}\sigma = e^{i(\pi m+\pi/2)}\sigma$, by (2.2). But we have just shown this is impossible. Thus S has no poles at $e^{i(m\pi+\pi/2)}\sigma$ for any $m \in \mathbb{Z} \setminus \{0\}$.

If P does not have eigenvalue $-\sigma^2$, then $R(\lambda)$ and $\Phi(\lambda)$ are both regular at $i\sigma$. Thus combining the first part of the theorem with Lemma 4.4 we see that $R(\lambda)$ is regular at $e^{i(m\pi+\pi/2)\sigma}$. □

Now we can see immediately why the distinction between (2.1) and (2.2) is so important here. If (2.1) were true, in even dimension d we would have that $I - S(i\sigma)$ is skew-adjoint, and hence has only imaginary eigenvalues. However, since it is rather that $\mathcal{R}(I - S(i\sigma))$ which is skew-adjoint, the question is more subtle. However, something can still be said.

Corollary 5.2. *Let d be even. Suppose $V \in L^\infty_0(\mathbb{R}^d;\mathbb{R})$ and $\mathcal{O} \subset \mathbb{R}^d$ is an open bounded set with smooth boundary. Let P be the operator $-\Delta + V$ on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$, with Dirichlet or Neumann boundary conditions. If P has no negative eigenvalues and if both $\{x \in \mathbb{R}^d : -x \in \mathcal{O}\} = \mathcal{O}$ and $V(-x) \equiv V(x)$, then P has no resonances with argument $\pi/2 + m\pi$, $m \in \mathbb{Z} \setminus \{0\}$.*

Proof. This follows immediately from Theorem 5.1 and Theorem 4.5 when combined with Corollary 2.4 which showed that in this setting $I - S(i\sigma)$ is skew adjoint. Note that we do not specify whether the purely imaginary resonances are resolvent resonances or scattering resonances, since the theorem and our assumptions guarantee that there are neither. □

The case of $m = -1$ (and thus also $m = 1$) of the following corollary is proved in [2, Theorem 4.4]. The results of [2] combined with Theorem 5.1 immediately give us more.

Corollary 5.3. *Let $\mathcal{O} \subset \mathbb{R}^d$ be an open bounded set with smooth boundary so that $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ is connected. Let P the the operator $-\Delta$ on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ with Dirichlet or Neumann boundary conditions. Then P has no resonances with argument $\pi/2 + m\pi$, $m \in \mathbb{Z}\setminus\{0\}$. If P is instead the operator $-\Delta$ on $\mathbb{R}^d\setminus\overline{\mathcal{O}}$ satisfying the Robin-type boundary condition*

$$fu + \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial(\mathbb{R}^d \setminus \overline{\mathcal{O}})$$

where ν is the outward normal and f is a non-negative C^1 function, then P has at most finitely many resonances (resolvent or scattering) with argument $\pi/2 + m\pi$ for each $m \in \mathbb{Z} \setminus \{0\}$.

Proof. We note first the absence of negative eigenvalues in this setting.

From [2, Theorem 3.5], for the Robin boundary condition for a fixed obstacle there is a σ_0 so that $i^{-(d-1)}(I - S(e^{i\pi/2}\sigma))\mathcal{R}$ is a negative operator for $\sigma \in (\sigma_0, \infty)$. From [13] or [2], for the Dirichlet (Neumann) boundary condition, $i^{-(d-1)}(I - S(e^{i\pi/2}\sigma))\mathcal{R}$ is positive (negative) for any $\sigma > 0$. From the results of [13, Section 4], the eigenvalues of $(I - S(e^{i\pi/2}\sigma))\mathcal{R}$ are purely imaginary, for $\sigma > \sigma_0$ for the Robin case, and all $\sigma > 0$ for the other cases. Then Theorem 5.1 finishes the proof, since we know from results of Vodev [32, 33] that there are only finitely many resonances on the interval $e^{i(\pi/2+m\pi)}(0, \sigma_0)$. □

6. Purely imaginary resolvent resonances for fixed sign potentials

In this section we prove Theorem 1.1 on resolvent resonances for Schrödinger operators with potentials $V \in L^\infty_0(\mathbb{R}^d)$ with fixed sign. Again in this section we assume d is even. It would be possible to prove a slightly weaker version of Theorem 1.1 in a manner analogous to the obstacle case, Corollary 5.3, invoking some results of [13, 31]. However, we choose to do this in a somewhat different way relying on the structure of the resolvent. This method has some advantages. For example, we prove that if $e^{i(m\pi+\pi/2)}\sigma$, $\sigma > 0$, $m \in \mathbb{Z}$ is a resonance of $-\Delta + V$ and $V \leq 0$, $V \in L^\infty_0(\mathbb{R}^d)$, then $-\sigma^2$ is an eigenvalue of $-\Delta + V$.

We use the notation $R_V(\lambda)$ for the meromorphic continuation of the resolvent of $-\Delta + V$, so that for $\lambda \in \Lambda_0$, $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$.

As in [5], we reduce the problem to studying an operator on Λ_0 using the following identity. For $\lambda \in \Lambda_0$, the point $e^{im\pi}\lambda \in \Lambda_m$. The resolvent $R_0(\lambda)$ of H_0 satisfies

$$R_0(e^{im\pi}\lambda) = R_0(\lambda) + imT(\lambda), \tag{6.1}$$

where the operator $T(\lambda)$ has the integral kernel:

$$T(x, y; \lambda) = \frac{1}{2}(2\pi)^{1-d}\lambda^{d-2} \int_{\mathbb{S}^{d-1}} e^{i\lambda\omega\cdot(x-y)}\, d\omega. \tag{6.2}$$

A crucial property of this operator is that for $d \geq 2$ even and $\chi \in L^\infty_0(\mathbb{R}^d; \mathbb{R})$, the operator $\chi T(i\sigma)\chi$ is self-adjoint for real $\sigma > 0$. To see this, note that from (6.2), we have

$$T(x, y; i\sigma) = \frac{(-1)^{d/2+1}}{2}(2\pi)^{1-d}\sigma^{d-2} \int_{\mathbb{S}^{d-1}} e^{-\sigma\omega\cdot(x-y)}\, d\omega \tag{6.3}$$

with $d/2 \in \mathbb{N}$. It follows, using the change of variable $\omega = -\omega'$, that $(\chi T(i\sigma)\chi)^* = \chi T(i\sigma)\chi$. We note that this is in contrast with the case of $d \geq 1$ *odd*, where the difference $R_0(i\sigma) - R_0(-i\sigma)$ is self-adjoint.

Proof of Theorem 1.1. As outlined above, we look for zeros of the form $\sigma_m = e^{i(m+1/2)\pi}\sigma$, where $\sigma > 0$ and $\sigma_m \in \Lambda_m$. We define the multiplication operator $\operatorname{sgn} V(x) = +1$ on $\{x \in \mathbb{R}^d \mid V(x) \geq 0\}$ and $\operatorname{sgn} V(x) = -1$ for $\{x \in \mathbb{R}^d \mid V(x) < 0\}$. Then the potential has the decomposition $V(x) = \operatorname{sgn} V(x)\ |V(x)|$. For $\lambda \in \Lambda_0$ we write the resolvent formula as

$$|V|^{1/2} R_V(\lambda)|V|^{1/2}(I + (\operatorname{sgn} V)|V|^{1/2} R_0(\lambda)|V|^{1/2}) = |V|^{1/2} R_0(\lambda)|V|^{1/2}, \tag{6.4}$$

so we study the operator

$$I + K(\lambda) = I + (\operatorname{sgn} V)|V|^{1/2} R_0(\lambda)|V|^{1/2}. \tag{6.5}$$

Since V has compact support, the operator $K(\lambda)$ has an analytic continuation to the Riemann surface Λ. It follows from this fact and (6.4) that the zeros of $I + K(\lambda)$ on the m^{th}-sheet Λ_m are the resonances of the operator H_V on Λ_m. The analytic continuation formula for the free resolvent (6.1) gives

$$\begin{aligned} K(e^{i\pi m}\lambda) =& (\operatorname{sgn} V)|V|^{1/2} R_0(e^{im\pi}\lambda)|V|^{1/2} \\ =& (\operatorname{sgn} V)|V|^{1/2} R_0(\lambda)|V|^{1/2} - im(\operatorname{sgn} V)|V|^{1/2} T(\lambda)|V|^{1/2}. \end{aligned} \tag{6.6}$$

In order to investigate purely imaginary poles on Λ_m, $m \in \mathbb{Z}^*$, we restrict to $\lambda = i\sigma = e^{i\pi/2}\sigma \in \Lambda_0$, with $\sigma > 0$, so that $e^{i(m\pi+\pi/2)}\sigma \in \Lambda_m$ is purely imaginary. We obtain from the reduction in (6.6)

$$K(ie^{im\pi}\sigma) = (\operatorname{sgn} V)|V|^{1/2} R_0(i\sigma)|V|^{1/2} - im(\operatorname{sgn} V)|V|^{1/2} T(i\sigma)|V|^{1/2}.$$

Since $R_0(i\sigma) = (-\Delta + \sigma^2)^{-1}$, $R_0(i\sigma)$ is self-adjoint and $R_0(i\sigma) > 0$. On the other hand, we showed that $i|V|^{1/2} T(i\sigma)|V|^{1/2}$ is skew-adjoint in the case of d even.

We now consider the operator $I + K(ie^{im\pi}\sigma)$ for potentials V having fixed sign. If $V \geq 0$, the operator $I + V^{1/2} R_0(i\sigma) V^{1/2}$ is strictly positive. On the other hand, the trace-class operator $iV^{1/2} T(i\sigma) V^{1/2}$ is skew-adjoint and therefore has only pure imaginary eigenvalues. Consequently, $I + K(e^{im\pi} i\sigma)$ has no zeros for $\sigma > 0$, meaning there are no purely imaginary zeros on any sheet. If $-V \geq 0$, the formula for $K(ie^{im\pi}\sigma)$ becomes:

$$K(ie^{im\pi}\sigma) = -|V|^{1/2} R_0(i\sigma)|V|^{1/2} + im|V|^{1/2} T(i\sigma)|V|^{1/2}.$$

The operator $I - |V|^{1/2} R_0(i\sigma)|V|^{1/2}$ has a zero if and only if $-\sigma^2$ is an eigenvalue of $-\Delta + V$, and the multiplicities agree. Because the trace-class operator $i|V|^{1/2} T(i\sigma)|V|^{1/2}$ is still skew-adjoint, there can be no zeros of $I + K(ie^{i\pi m}\sigma)$ unless $-\sigma^2$ is an eigenvalue of $-\Delta + V$. Consequently, if N_V denotes the number of negative eigenvalues of $-\Delta + V$, there can be at most $N_V < \infty$ purely imaginary resonances on any sheet Λ_m, $m \in \mathbb{Z}^*$. □

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