ADVANCES IN MATHEMATICS 7, 145–210 (1971)

R-Torsion and the Laplacian on Riemannian Manifolds

D. B. RAY AND I. M. SINGER*

Department of Mathematics, Massachusetts, Institute of Technology, Cambridge, Massachusetts 02139

INTRODUCTION

Let W be a compact oriented Riemannian manifold of dimension N, and let K be a simplicial complex which is a smooth triangulation of W. The Reidemeister–Franz torsion (or R-torsion) τ of K is a function of certain representations of the fundamental group of K. Since it is a combinatorial invariant, and since smooth triangulations of W are equivalent, this torsion is a manifold invariant.

We raise the question as to how to describe this manifold invariant in analytic terms. Arnold Shapiro once suggested that there might be a formula for the torsion in terms of the Laplacian Δ acting on differential forms on W. Our candidate T involves the zeta function for appropriate Laplacians. Though we have been unable to prove that $T = \tau$, we show in this paper that T is a manifold invariant and present some evidence that $T = \tau$.

If one thinks of analytic torsion as an invariant associated to the De Rham complex, it is natural to ask whether there are analogous invariants for other elliptic complexes. For complex manifolds and the $\bar{\partial}$-complex, there is indeed such a holomorphic invariant, which will be the subject of a subsequent paper.

In Section 1 we give a short exposition of Reidemeister–Franz torsion and motivate our definition of the analytic torsion T. In Section 2 are collected the main results of the paper. First we prove that $T = T_W$ is independent of the metric of W, for W closed. Next we prove three results which are formal analogs of known properties of the Reidemeister–Franz torsion, namely, $T_W = 1$ if W is closed and has even dimension; $T_{W_1 \times W_2} = (T_{W_1})^{\chi(W_2)}$, $\chi(W_2)$ being the Euler characteristic of W_2, if

* The research of the first author was supported in part by National Science Foundation grant GP 13778; that of the second by National Science Foundation grant GP 13876 and by a grant from the Guggenheim Foundation.

W_2 is simply connected; and, finally, if W_1 and W_2 have the same universal covering manifold and if the fundamental group $\pi_1(W_1)$ of W_1 is a subgroup of that of W_2, then $T_{W_1} = T_{W_2} \circ U$, where U carries a representation of $\pi_1(W_1)$ into the induced representation of $\pi_1(W_2)$.

We also describe in Section 2 a possible method of proof that T_W/τ_W is constant on the representations of $\pi_1(W)$ for which the torsion is defined. Part of this program involves extending the definition of torsion to the case of Riemannian manifolds with boundary and with nontrivial homology. This is done in Section 3. To define τ in this case we must choose a base for the homology classes of W, which we get by the Hodge theorem from an orthonormal base of harmonic forms.

In Section 4 we present the de Rham–Hodge theory for manifolds with boundary, and prove that the combinatorial torsion defined in Section 3 is invariant under subdivision, hence independent of the triangulation of W used to define it. At the heart of the proof is a relation, proved by Kodaira [4] for closed manifolds, between duality in the differential form setting and in the homology-cohomology setting.

In Section 5, some results about the heat equation are described, including the Hodge theorem for manifolds with boundary. In Section 6 we compute the behavior of the trace of the heat kernel as the metric changes.

Finally, in Section 7 we apply the results of the four preceding sections to prove the following. Suppose a manifold W with boundary is equipped with two metrics which determine the same normal direction at the boundary. Suppose O_1 and O_2 are two representations of the fundamental group $\pi_1(W)$, for which the torsion is defined. Then the differences $\log(T_W(O_1)/\tau_W(O_1)) - \log(T_W(O_2)/\tau_W(O_2))$ are the same for both metrics.

1. The Analytic Torsion

Let us begin with a description of Reidemeister–Franz torsion. We will follow Milnor [10, Section 8] and in particular use his definition of R-torsion, which differs from that of Reidemeister and Franz.

If V is a finite-dimensional vector space over the reals and if $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ are two bases for V, let $[\mathbf{w}/\mathbf{v}]$ denote $|$ determinant T $|$, where T is the matrix representing the change of base from $\mathbf{v}$ to $\mathbf{w}$: $w_i = \sum t_{ij} v_j$.

Suppose

$$C : C_N \xrightarrow{\partial} C_{N-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0$$

is a chain complex of finite real modules. As usual, let Z_q denote the kernel of ∂ in C_q, $B_q \subset Z_q$ the image of C_{q-1} under ∂, and $H_q(C) = Z_q/B_q$ the q-th homology group of C.

Suppose we are given preferred bases $\mathbf{c}_q$ for C_q and $\mathbf{h}_q$ for $H_q(C)$ for each q. Choose a base $\mathbf{b}_q$ for B_q for each q; let $\tilde{\mathbf{b}}_{q-1}$ be an independent set in C_q such that $\partial\tilde{\mathbf{b}}_{q-1} = \mathbf{b}_{q-1}$, and $\tilde{\mathbf{h}}_q$ an independent set in Z_q representing the base $\mathbf{h}_q$. Then $(\mathbf{b}_q, \tilde{\mathbf{h}}_q, \tilde{\mathbf{b}}_{q-1})$ is a base for C_q. Since, clearly, $[\mathbf{b}_q \tilde{\mathbf{h}}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q]$ depends only on $\mathbf{b}_q, \mathbf{h}_q, \mathbf{b}_{q-1}$, we denote it by $[\mathbf{b}_q, \mathbf{h}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]$.

DEFINITION 1.1. $\tau(C)$ is the positive real number defined by

$$\log \tau(C) = \sum_{q=0}^{N} (-1)^q \log[\mathbf{b}_q, \mathbf{h}_q, \mathbf{b}_{q-1}/\mathbf{c}_q].$$

Remark. It is easy to see that $\tau(C)$ does not depend on the choice of the bases $\mathbf{b}_q$ for the B_q : if $\mathbf{b}_q'$ is another choice, then

$$[\mathbf{b}_q', \mathbf{h}_q, \mathbf{b}_{q-1}'/\mathbf{c}_q] = [\mathbf{b}_q'/\mathbf{b}_q][\mathbf{b}_{q-1}'/\mathbf{b}_{q-1}][\mathbf{b}_q, \mathbf{h}_q, \mathbf{b}_{q-1}/\mathbf{c}_q],$$

and the first two factors on the right drop out in the formula for $\tau(C)$.

The R-torsion arises in the following context. Let K be a finite-cell complex and $\tilde{K}$ the simply connected covering space of K with the fundamental group π_1 of K acting as deck transformations on $\tilde{K}$. Think of K embedded as a fundamental domain in $\tilde{K}$, so that $\tilde{K}$ is just the set of translates of K under π_1. In this way, the real chain groups $C_q(\tilde{K})$ become modules over the real group algebra $R(\pi_1)$, with a preferred base consisting of the cells of K.

Let e be a cell of K; its boundary ∂e in $\tilde{K}$ will not in general be contained in K, but will be a combination of translates of cells of K by deck transformations. Hence relative to the preferred base, the boundary operator on the $R(\pi_1)$-module $C_q(\tilde{K})$ is a matrix with coefficients in $R(\pi_1)$.

Now let O be a representation of $\pi_1(K)$ by orthogonal $n \times n$ matrices. We may think of O as making R^n a right $R(\pi_1)$-module. Define the chain complex $C(K, O)$ by

$$C_q(K, O) = R^n \underset{R(\pi_1)}{\otimes} C_q(\tilde{K}). \tag{1.2}$$

$C_q(K, O)$ is a real vector space, and we can choose a preferred base $(x_i \otimes e_j)$, where x_i runs through an orthogonal base of R^n and e_j through the preferred base of $C_q(\tilde{K})$ consisting of the cells of K.

DEFINITION 1.3. Let O be a representation of the fundamental group π_1 by orthogonal matrices for which $C(K, O)$ is acyclic. The R-torsion is defined for such a representation by

$$\tau_K(O) = \tau(C(K, O)),$$

where $C(K, O)$ has the preferred base described above. (Since $H(C(K, O))$ is assumed to be zero, no homology base occurs in the definition of $\tau(C(K, O))$.)

Remark. The preferred base of $C(K, O)$ depends on an arbitrary embedding of K in the covering space $\tilde{K}$. A different choice of the embedding, however, produces a new base related to the old one by an elementary matrix whose entries are group elements. Since this corresponds to a change of base in $C(K, O)$ by an orthogonal matrix, the R-torsion τ_K is independent of this arbitrary choice. Similarly, τ_K does not depend on the arbitrary choice of the orthonormal base $\mathbf{x}$ of R^n.

It is known [15; 10, Section 7] that τ_K is a combinatorial invariant of K. Hence if W is a compact oriented manifold, any smooth triangulation of W gives the same R-torsion, which we denote by τ_W.

We will now define the analytic torsion. For the present, W will be a compact oriented manifold without boundary, of dimension N. Given a representation O of the fundamental group $\pi_1(W)$ by orthogonal matrices, let $E(O)$ be the associated vector bundle, and let $\mathscr{D} = \sum \mathscr{D}^q$ be the linear space of C^∞ differential forms on W with values in $E(O)$. $\mathscr{D}^q$ is the space of C^∞ sections of the sheaf $\Lambda^q \otimes E(O)$, where Λ^q is the de Rham sheaf. We have the usual exterior differential $d : \mathscr{D}^q \to \mathscr{D}^{q+1}$, with $d^2 = 0$.

Suppose that W has a Riemannian metric. This defines a duality $* : \mathscr{D}^q \to \mathscr{D}^{N-q}$ and provides $\mathscr{D}^q$ with an inner product

$$(f, g) = \int_W f \wedge *g,$$

where $\wedge : \{\Lambda^q \otimes E(O), \Lambda^p \otimes E(O)\} \to \Lambda^{q+p}$ is the map determined by the usual exterior product of differential forms and the inner product in $E(O)$. With this inner product, $\mathscr{D}$ becomes a Hilbert space, and $\delta = (-1)^{Nq+N+1} * d *$ is the formal adjoint of d on $\mathscr{D}^q$.

The Laplacian

$$\Delta = -(\delta d + d\delta) \tag{1.4}$$

is symmetric and negative-definite on $\mathscr{D}$, and is known [7] to have a pure point spectrum. We write Δ_q for the restriction of Δ to $\mathscr{D}^q$.

Suppose now that zero is not an eigenvalue of Δ, corresponding by the Hodge theorem to the assumption in Definition 1.3 that $C(K, O)$ is acyclic. Then the zeta function $\zeta_{q,O}$ of Δ_q on $\mathscr{D}^q$ is defined by

$$\begin{aligned} \zeta_{q,O}(s) &= \sum (-\lambda_n)^{-s} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(e^{t\Delta_q})\, dt \end{aligned} \tag{1.5}$$

for $\operatorname{Re}(s)$ large, the sum running over the eigenvalues λ_n of Δ_q. It is known [16] that $\zeta_{q,O}$ extends to a meromorphic function of s which is analytic at $s = 0$.

Definition 1.6. Let O be a representation of $\pi_1(W)$ by orthogonal matrices such that Δ is strictly negative on $\mathscr{D}(W, O)$. The analytic torsion $T_W(O)$ is defined for such a representation as the positive real root of

$$\log T_W(0) = \tfrac{1}{2} \sum_{q=0}^{N} (-1)^q q \zeta'_{q,O}(0).$$

This is our candidate for the analytic torsion, and we shall now explain where the formula comes from.

First we note that the R-torsion of a smooth triangulation K of W can be expressed in terms of determinants formed from the boundary operator ∂ on $C(K, O)$, as follows. The choice of a preferred base for each C_q represents $\partial : C_q \to C_{q-1}$ as a real matrix. Let $\partial^* : C_q \to C_{q+1}$ be the transpose matrix, and define the combinatorial Laplacian $\Delta^{(c)} : C_q \to C_q$ by

$$\Delta^{(c)} = -(\partial^*\partial + \partial\partial^*).$$

Let $\Delta_q^{(c)}$ be the matrix representing $\Delta^{(c)}$ on each C_q. Under the assumption that $C(K, O)$ is acyclic, $\Delta_q^{(c)}$ is nonsingular for each q.

PROPOSITION 1.7. *If the R-torsion $\tau_K(O)$ is defined for the representation O, then*

$$\log \tau_K(O) = \tfrac{1}{2} \sum_{q=0}^{N} (-1)^{q+1} q \log \det(-\Delta_q^{(c)}).$$

Proof. On each chain group $C_q(K, O)$, the preferred base determines an inner product in which the combinatorial Laplacian is a symmetric and strictly negative matrix. Moreover, (since $\partial^2 = 0$), the subspace $B_q = \partial(C_{q+1})$ of boundaries is invariant under $\Delta_q^{(c)}$. Let $\mathbf{b}_q = (b_q^{\,1},\ldots, b_q^{r_q})$ be an orthonormal base for B_q consisting of eigenvectors of $\Delta_q^{(c)}$:

$$\Delta_q^{(c)} b_q^{\,j} = -\partial\partial^* b_q^{\,j} = \lambda_{q,j} b_q^{\,j}.$$

Having chosen this base for each B_q, set

$$\tilde{b}_{q-1}^j = -\frac{1}{\lambda_{q-1,j}} \partial^* b_{q-1}^j, \qquad j = 1,\ldots, r_{q-1},$$

so that $\partial \tilde{b}_{q-1}^j = b_{q-1}^j$. Note that the vectors $\tilde{b}_{q-1}^j$ are orthogonal, with

$$\| \tilde{b}_{q-1}^j \|^2 = \frac{1}{\lambda_{q-1,j}^2} (\partial^* b_{q-1}^j, \partial^* b_{q-1}^j) = -\frac{1}{\lambda_{q-1,j}}.$$

Hence, assuming $C(K, O)$ acyclic,

$$\begin{aligned} \log \tau_K(O) &= \sum_{q=0}^{N} (-1)^q \log[\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q] \\ &= \sum_{q=1}^{N} (-1)^q \log \prod_{1}^{r_{q-1}} (-\lambda_{q-1,j})^{-1/2}. \end{aligned}$$

But the orthonormal base of C_q consisting of $b_q^{\,j}$, $j = 1,\ldots, r_q$, and $(-\lambda_{q-1,j})^{1/2}\tilde{b}_{q-1}^j$, $j = 1,\ldots, r_{q-1}$, clearly diagonalizes $\Delta_q^{(c)}$, so that

$$\det(-\Delta_q^{(c)}) = \prod_{1}^{r_q} (-\lambda_{q,j}) \prod_{1}^{r_{q-1}} (-\lambda_{q-1,j}).$$

This implies

$$\log \prod_{1}^{r_{q-1}} (-\lambda_{q-1,j}) = \sum_{k=q}^{N} (-1)^{k-q} \log \det(-\Delta_k^{(c)}),$$

which transforms the above expression for $\log \tau_K(O)$ into that given in Proposition 1.7.

Next, observe that the determinant of the nonsingular matrix $-\Delta_q^{(c)}$ can be expressed in terms of its zeta function

$$\zeta_q^{(c)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(e^{t\Delta_q^{(c)}})\, dt$$

$$= \operatorname{Tr}(-\Delta_q^{(c)})^{-s}.$$

Indeed, since for each eigenvalue λ of $\Delta_q^{(c)}$,

$$\frac{d}{ds}(-\lambda)^{-s} = -\log(-\lambda)(-\lambda)^{-s},$$

we have $\log \det(-\Delta_q^{(c)}) = -\zeta_q^{(c)\prime}(0)$. Thus the formula for the analytic torsion is a formal analog of that for the R-torsion. In view of the relation between the complexes $\mathscr{D}(W, O)$ and $C(K, O)$ given by the de Rham theorem, we are led to believe that the analytic torsion and the R-torsion may in fact be equal.

2. Some Properties of the Analytic Torsion

In this section we prove four properties of the analytic torsion. Three of these are formal analogs of known properties of the R-torsion, and the proofs depend primarily on formal manipulations. The proof of Theorem 2.1, on the other hand, requires some properties of the heat kernel, given in Sections 5 and 6.

We also outline an argument which might prove that for two representations O_1 and O_2 of $\pi_1(W)$, $T_W(O_1)/\tau_W(O_1) = T_W(O_2)/\tau_W(O_2)$. The remainder of the paper carries out part of this program.

Our first task in this section is to prove that the analytic torsion T_W does not depend on the arbitrary choice of a Riemannian metric for W.

Theorem 2.1. *Let W be a compact oriented manifold without boundary, and let O be a representation of the fundamental group $\pi_1(W)$ by orthogonal matrices with the property that the cohomology with coefficients in the associated vector bundle $E(O)$ is trivial. Then $T_W(O)$ has the same value for any choice of a Riemannian metric on W.*

Proof. We will prove in Theorem 2.3 that $\log T_W = 0$ when W has even dimension. Since that proof does not make use of Theorem 2.1, we may assume here that $N = \dim W$ is odd.

Suppose ρ_0 and ρ_1 are two Riemannian metrics on W. Set $\rho_u = (1 - u)\rho_0 + u\rho_1$, and let $\Delta_q(u)$ denote the Laplacian on $\mathscr{D}^q(W, O)$ formed with the metric ρ_u. By the Hodge theorem, $\Delta_q(u)$ is strictly negative under the assumption that the cohomology of W with coefficients in $E(O)$ is trivial. Hence

$$f(u, s) = \tfrac{1}{2} \sum_{q=0}^{N} (-1)^q q \int_0^\infty t^{s-1} \operatorname{Tr}(e^{t\Delta_q(u)})\, dt$$

defines a function of s for $\operatorname{Re} s$ sufficiently large, which as remarked after (1.5) extends to a meromorphic function in the s-plane. According to (1.5) and Definition 1.6, $\log T_W(O) = f(u, 0)$ for the metric ρ_u on W, so we have to show that $(\partial/\partial u) f(u, 0) = 0$.

By Proposition 6.1,

$$\frac{\partial}{\partial u} \operatorname{Tr}(e^{t\Delta_q(u)}) = t \operatorname{Tr}(e^{t\Delta_q(u)} \dot{\Delta}_q)$$

where $\dot{\Delta}_q = \alpha\delta d - \delta\alpha d + d\alpha\delta - d\delta\alpha$, α being the algebraic operator $\alpha = *^{-1}\dot{*} = *^{-1}(\partial/\partial u\, *)$ on $\mathscr{D}^q$. Clearly,

$$\operatorname{Tr}(e^{t\Delta_q(u)}) \leqslant Ce^{-\epsilon t}, \qquad t \geqslant t_0 > 0,$$

for $C, \epsilon > 0$ independent of u in $[0, 1]$, so that we can differentiate under the integral to get

$$\frac{\partial}{\partial u} f(u, s) = \tfrac{1}{2} \sum_{q=0}^{N} (-1)^q q \int_0^\infty t^s \operatorname{Tr}(e^{t\Delta_q(u)} \dot{\Delta}_q)\, dt \tag{2.2}$$

for $\operatorname{Re} s$ large.

We now compute $\operatorname{Tr}(e^{t\Delta_q(u)}\dot{\Delta}_q)$. If A is of trace class and B is a bounded operator, it is well known that $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$. Hence

$$\begin{aligned}
\operatorname{Tr}(e^{t\Delta_q}\alpha\delta d) &= \operatorname{Tr}(e^{\frac{1}{2}t\Delta_q}\alpha\delta d e^{\frac{1}{2}t\Delta_q}) \\
&= \operatorname{Tr}(\delta d e^{t\Delta_q}\alpha), \\
\operatorname{Tr}(e^{t\Delta_q}\delta\alpha d) &= \operatorname{Tr}(d e^{t\Delta_q}\delta\alpha), \\
\operatorname{Tr}(e^{t\Delta_q} d\alpha\delta) &= \operatorname{Tr}(\delta e^{t\Delta_q} d\alpha).
\end{aligned}$$

Since $d\Delta_q = \Delta_{q+1}d$ and $\delta\Delta_q = \Delta_{q-1}\delta$, we have

$$\operatorname{Tr}(e^{t\Delta_q}\dot{\Delta}_q) = \operatorname{Tr}(e^{t\Delta_q}\delta d\alpha) - \operatorname{Tr}(e^{t\Delta_{q+1}}d\delta\alpha) + \operatorname{Tr}(e^{t\Delta_{q-1}}\delta d\alpha) - \operatorname{Tr}(e^{t\Delta_q}d\delta\alpha).$$

Thus

$$\begin{aligned}\sum_{q=0}^{N}(-1)^q\, q\operatorname{Tr}(e^{t\Delta_q}\dot{\Delta}_q) &= \sum_{q=0}^{N}(-1)^{q+1}(\operatorname{Tr}(e^{t\Delta_q}\delta d\alpha) + \operatorname{Tr}(e^{t\Delta_q}d\delta\alpha))\\ &= \sum_{q=0}^{N}(-1)^q\operatorname{Tr}(e^{t\Delta_q}\Delta_q\alpha)\\ &= \frac{d}{dt}\sum_{q=0}^{N}(-1)^q\operatorname{Tr}(e^{t\Delta_q}\alpha).\end{aligned}$$

Using this in (2.2),

$$\begin{aligned}\frac{\partial}{\partial u}f(u,s) &= \tfrac{1}{2}\sum_{q=0}^{N}(-1)^q\int_0^\infty t^s\frac{d}{dt}\operatorname{Tr}(e^{t\Delta_q(u)}\alpha)\,dt\\ &= \tfrac{1}{2}\,s\sum_{q=0}^{N}(-1)^{q+1}\int_0^\infty t^{s-1}\operatorname{Tr}(e^{t\Delta_q(u)}\alpha)\,dt.\end{aligned}$$

The last equality is obtained by integration by parts. The integrated terms vanish for Re s large because $\operatorname{Tr}(e^{t\Delta_q(u)}\alpha)$ decreases exponentially for large t and is $O(t^{-N/2})$ for small t.

To complete the proof of Theorem 2.1 we need only show that the meromorphic extension of the function

$$g(u,s) = \int_0^\infty t^{s-1}\operatorname{Tr}(e^{t\Delta_q(u)}\alpha)\,dt$$

has no pole at the origin.

This, however, follows for W of odd dimension by a straightforward extension of the results of Minakshisundaram and Pleijel [11] (see (5.6); Seeley [16] has proved a much broader generalization), namely, for Re s large, $(-\Delta_q)^{-s}$ is an integral operator with kernel

$$K_s(x,y) \in \hom(\mathscr{D}^q(y), \mathscr{D}^q(x));$$

for each x in W, the map $s \to K_s(x,x)$ extends to a meromorphic function in the s-plane, which vanishes at $s = 0$ when dim W is odd.

Now $(-\Delta_q)^{-s}\alpha$ has the kernel $K_s(x, y)\,\alpha(y)$ for $\operatorname{Re} s$ large, and the extension of $K_s(x, x)\,\alpha(x)$ also vanishes at $s = 0$. Hence

$$\operatorname{Tr}((-\Delta_q)^{-s}\,\alpha) = \int_W \operatorname{Tr}(K_s(x, x)\,\alpha(x))$$

has a meromorphic extension vanishing at $s = 0$.

But standard calculations give

$$g(u, s) = \int_0^\infty t^{s-1} \operatorname{Tr}(e^{t\Delta_q(u)}\alpha)\,dt$$
$$= \Gamma(s) \operatorname{Tr}((-\Delta_q(u))^{-s}\,\alpha),$$

and so $g(u, s)$ is regular at $s = 0$.

We next prove three properties of the analytic torsion which reflect known properties of the R-torsion.

THEOREM 2.3. *Suppose W is an oriented compact manifold without boundary, of even dimension. Then* $\log T_W(O) = 0$. (*That* $\log \tau_W = 0$ *in this case is proved in* [8].)

Proof. Let O be a representation of $\pi_1(W)$ by orthogonal matrices for which the Laplacian Δ is strictly negative on $\mathscr{D}(W, O)$. We will show, using duality, that

$$\sum_{q=0}^{N} (-1)^q\, q\zeta_{q,O}(s) = 0,$$

where $\zeta_{q,O}$ is the zeta function defined in (1.5).

Let λ be an eigenvalue of Δ_q, and let $\mathscr{E}_q(\lambda)$ be the subspace of $\mathscr{D}^q(W, O)$ consisting of the eigenforms belonging to λ. We have assumed that zero is not an eigenvalue, so we may define the two maps

$$A_q'(\lambda) = -\lambda^{-1}\,d\delta,$$
$$A_q''(\lambda) = -\lambda^{-1}\,\delta d,$$

on $\mathscr{E}_q(\lambda)$. Since $(d\delta)^2 = -d\delta\Delta$, $(\delta d)^2 = -\delta d\Delta$, A_q' and A_q'' are orthogonal projections of $\mathscr{E}_q(\lambda)$ onto the two subspaces, respectively,

$$\mathscr{E}_q'(\lambda) = \{\phi \in \mathscr{E}_q(\lambda),\, d\phi = 0\},$$
$$\mathscr{E}_q''(\lambda) = \{\phi \in \mathscr{E}_q(\lambda),\, \delta\phi = 0\}.$$

Moreover, the equation

$$\phi = \lambda^{-1} \Delta\phi = -\lambda^{-1}(d\delta\phi + \delta d\phi)$$

shows that $\mathscr{E}_q(\lambda)$ is the direct sum of the subspaces $\mathscr{E}_q'(\lambda)$, $\mathscr{E}_q''(\lambda)$. Finally, the map $(-\lambda)^{-1/2}d$ is an isometry of $\mathscr{E}_q''(\lambda)$ onto $\mathscr{E}_{q+1}'(\lambda)$, with inverse $(-\lambda)^{-1/2}\delta$.

Let $N_q'(\lambda)$ and $N_q''(\lambda)$ be the dimensions of the spaces $\mathscr{E}_q'(\lambda)$ and $\mathscr{E}_q''(\lambda)$. Because of the above, the multiplicity of λ is $N_q(\lambda) = N_q'(\lambda) + N_q''(\lambda) = N_q'(\lambda) + N_{q+1}'(\lambda)$. Hence we may write the zeta function of (1.5) as

$$\begin{aligned}\zeta_{q,O}(s) &= \sum (-\lambda)^{-s} N_q(\lambda) \\ &= \sum (-\lambda)^{-s}(N_q'(\lambda) + N_{q+1}'(\lambda)) \\ &= \sum (-\lambda)^{-s}(N_q''(\lambda) + N_{q-1}''(\lambda)),\end{aligned}$$

yielding

$$\sum_{q=0}^{N} (-1)^q q\zeta_{q,O}(s) = \sum_{q=1}^{N} (-1)^q \sum (-\lambda)^{-s} N_q'(\lambda) \tag{2.4}$$

$$= \sum_{q=0}^{N-1} (-1)^q \sum (-\lambda)^{-s} N_q''(\lambda).$$

Finally, the duality operator $* : \mathscr{D}^q \to \mathscr{D}^{N-q}$, satisfying $*d\delta = \delta d*$, defines an isometry of $\mathscr{E}_q'(\lambda)$ onto $\mathscr{E}_{N-q}''(\lambda)$. Hence $N_q'(\lambda) = N_{N-q}''(\lambda)$, which upon substitution in (2.5) shows that

$$\sum_{q=0}^{N} (-1)^q q\zeta_{q,O}(s) = 0$$

when N is even.

THEOREM 2.5. *Suppose W_1 and W_2 are oriented compact manifolds without boundary, and suppose W_2 is simply connected. Then the analytic torsion of the product manifold $W_1 \times W_2$ is given by*

$$\log T_{W_1 \times W_2} = \chi(W_2) \log T_{W_1} .$$

where $\chi(W_2)$ *is the Euler characteristic of* W_2. (*See* [5] *for the corresponding result for R-torsion.*)

Proof. By Theorem 2.1, we are free to choose the metrics arbitrarily on the three manifolds. So we can assume that $W_1 \times W_2$ has the product metric of those of W_1 and W_2.

Let O be a representation of the fundamental group $\pi_1(W_1 \times W_2)$ by orthogonal matrices. Since $\pi_1(W_1 \times W_2) = \pi_1(W_1)$, the associated vector bundle $E(O, W_1 \times W_2)$ is the same as the vector bundle $E(O, W_1)$ lifted to $W_1 \times W_2$ via the projection map of $W_1 \times W_2$ onto W_1.

Suppose $f_1 \in \mathscr{D}^p(W_1, O)$ and $f_2 \in \mathscr{D}^q(W_2)$, the latter space consisting of real-valued C^∞ forms on W_2. Let $f_1 \otimes f_2$ be the wedge product of f_1 and f_2, lifted to $W_1 \times W_2$. Such forms, for $p + q = r$, span $\mathscr{D}^r(W_1 \times W_2, O)$. Clearly

$$d(f_1 \otimes f_2) = (df_1) \otimes f_2 + (-1)^p f_1 \otimes (df_2),$$

and, since the metric on $W_1 \times W_2$ is the product,

$$\delta(f_1 \otimes f_2) = (\delta f_1) \otimes f_2 + (-1)^p f_1 \otimes (\delta f_2).$$

Using these in the definition of the Laplacian $\Delta = -d\delta - \delta d$,

$$\Delta(f_1 \otimes f_2) = (\Delta f_1) \otimes f_2 + f_1 \otimes (\Delta f_2).$$

Hence if f_1 and f_2 are eigenforms of the Laplacian, with eigenvalues λ_1 and λ_2, then $f_1 \otimes f_2$ is an eigenform with eigenvalue $\lambda_1 + \lambda_2$. Since the forms $f_1 \otimes f_2$ span $\mathscr{D}(W_1 \times W_2, O)$, all eigenforms of the Laplacian on $\mathscr{D}(W_1 \times W_2, O)$ are obtained in this way.

Suppose now that for each q, zero is not an eigenvalue of the Laplacian on $\mathscr{D}^q(W_1, O)$. Then the same is true of $\mathscr{D}^q(W_1 \times W_2, O)$, and $T_{W_1}(O)$ and $T_{W_1 \times W_2}(O)$ are both defined. Let $N_p(\lambda, W_1)$ and $N_q(\mu, W_2)$ denote, respectively, the multiplicities of the eigenvalues λ and μ of the Laplacian on the spaces $\mathscr{D}^p(W_1, O)$ and $\mathscr{D}^q(W_2)$. Then according to the preceding paragraph, the zeta function of the Laplacian on $\mathscr{D}^r(W_1 \times W_2, O)$ is given for Re s large by

$$\zeta_{r,O,W_1 \times W_2}(s) = \sum_{\lambda,\mu} \sum_{p+q=r} (-\lambda - \mu)^{-s} N_p(\lambda, W_1)\, N_q(\mu, W_2),$$

and the alternating sum in the definition of the analytic torsion is

$$\sum_{r=0}^{N_1N_2} (-1)^r r\zeta_{r,O,W_1\times W_2}(s)$$

$$= \sum_{\lambda,\mu} (-\lambda-\mu)^{-s} \sum_{p=0}^{N_1} \sum_{q=0}^{N_2} (-1)^{p+q}(p+q)\, N_p(\lambda, W_1)\, N_q(\mu, W_2)$$

$$= \sum_{\lambda,\mu} (-\lambda-\mu)^{-s} \Big(\sum_{p=0}^{N_1} (-1)^p pN_p(\lambda, W_1)\Big)\Big(\sum_{q=0}^{N_2} (-1)^q N_q(\mu, W_2)\Big)$$

$$+ \sum_{\lambda,\mu} (-\lambda-\mu)^{-s} \Big(\sum_{p=0}^{N_1} (-1)^p N_p(\lambda, W_1)\Big)\Big(\sum_{q=0}^{N_2} (-1)^q qN_q(\mu, W_2)\Big).$$

(N_1 and N_2 denote, of course, the dimensions of the manifolds W_1 and W_2.)

We now follow the path which led to formula (2.4). If $\lambda \neq 0$, let $N_p'(\lambda, W_1)$ denote the dimension of the space

$$\begin{aligned}\mathscr{E}'(\lambda, W_1) &= \{\phi \in \mathscr{E}_p(\lambda, W_1), d\phi = 0\}\\ &= \{\phi \in \mathscr{D}^p(O, W_1), d\delta\phi = \lambda\phi\}.\end{aligned}$$

We have, as before, $N_p(\lambda, W_1) = N_p'(\lambda, W_1) + N'_{p+1}(\lambda, W_1)$; but this implies

$$\sum_{p=0}^{N_1} (-1)^p N_p(\lambda, W_1) = 0,$$

and the second sum on the right above vanishes. Similarly,

$$\sum_{q=0}^{N_2} (-1)^q N_q(\mu, W_2) = 0$$

for each nonzero eigenvalue μ of $\varDelta$ on $\mathscr{D}(W_2)$.

Hence

$$\sum_{r=0}^{N_1N_2} (-1)^r r\zeta_{r,O,W_1\times W_2}(s)$$

$$= \sum (-\lambda)^{-s} \Big(\sum_{p=0}^{N_1} (-1)^p pN_p(\lambda, W_1)\Big)\Big(\sum_{q=0}^{N_2} (-1)^q N_q(0, W_2)\Big).$$

By the Hodge theorem, $N_q(O, W_2)$ equals the q-th betti number of W_2, so that

$$\sum_{r=0}^{N_1 N_2} (-1)^r r\zeta_{r,O,W_1\times W_2}(s) = \chi(W_2) \sum_{p=0}^{N_1} (-1)^p p\zeta_{p,O,W_1}(s).$$

This identity has been proved for Re s sufficiently large, but of course it holds throughout the s-plane for the meromorphic extensions of the zeta functions. In particular, it implies the relation given in Theorem 2.5 for the analytic torsion.

The last of the three formal results involves the notion of induced representation [6]. Suppose G_2 is a group, and G_1 is a subgroup of finite index r in G_2. Suppose O is a representation of G_1 by $n \times n$ matrices. Let V be the space of maps ϕ of G_2 into R^n which satisfy

$$\phi(g_1 g) = O(g_1)\phi(g)$$

for g_1 in G_1. V is a real linear space of dimension nr, and the induced representation U^O of G_2 is defined on V by right translation:

$$U^O(g_2)\phi(g) = \phi(gg_2), \qquad g_2 \in G_2 .$$

THEOREM 2.6. *Suppose W_1 and W_2 are oriented compact manifolds without boundary, with the same universal covering manifold $\tilde{W}$. Suppose the fundamental group $\pi_1(W_1)$ is a subgroup of $\pi_1(W_2)$. Then the analytic torsions satisfy*

$$T_{W_1} = T_{W_2} \circ U,$$

where U carries a representation O of $\pi_1(W_1)$ into the induced representation U^O of $\pi_1(W_2)$. (A similar identity for R-torsion is given in [13].)

Remark. The fact that W_2 is compact clearly implies that $\pi_1(W_1)$ must have finite index r in $\pi_1(W_2)$.

Proof. By Theorem 2.1, we can assume that W_1 and W_2 have Riemannian metrics which lift to the same metric on the common universal covering space $\tilde{W}$. Let O be a representation of $\pi_1(W_1)$ by $n \times n$ orthogonal matrices, and $E(O)$ the corresponding vector bundle on W_1. We will establish an isomorphism between $\mathscr{D}^q(W_1, O)$ and $\mathscr{D}^q(W_2, U^O)$ which commutes with the Laplacian. Thus the spectrum of the Laplacian is the same on the two spaces, and $T_{W_2}(U^O)$ is defined and equals $T_{W_1}(O)$, whenever the latter is defined.

It will be convenient to consider besides the representation U^O an equivalent representation defined as follows: Write G_1 for $\pi_1(W_1)$ and $G_2 = \bigcup_{k=1}^{r} G_1\alpha_k$ for $\pi_1(W_2)$, where the α_k represent the cosets G_2/G_1, with $\alpha_1 = e$. Since a map ϕ in V is determined by its values on the α_k, $T\phi = \sum_{k=1}^{r} \oplus \phi(\alpha_k)$ defines an isomorphism of V onto the direct sum R^{nr} of r copies of R^n. Let $\dot{O}(g) = O(g)$ if $g \in G_1$, $\dot{O}(g) = 0$ if $g \notin G_1$, and define the representation ρ of G_2 on R^{nr} by

$$\rho(g)\left(\sum \oplus y_k\right) = \sum_{k=1}^{r} \oplus \sum_{j=1}^{r} \dot{O}(\alpha_k g \alpha_j^{-1})\, y_j\,.$$

Then ρ and U^O are equivalent; for if $\phi \in V$, then

$$\begin{aligned} TU^O(g)\,\phi &= \sum \oplus\, U^O(g)\,\phi(\alpha_k) \\ &= \sum \oplus\, \phi(a_k g) \\ &= \rho(g)\left(\sum \oplus\, \phi(\alpha_j)\right) \\ &= \rho(g)\, T\phi, \end{aligned}$$

since each $\alpha_k g$ is uniquely expressible as $g_1\alpha_j$ for $g_1 = \alpha_k g \alpha_j^{-1}$ in G_1.

Now the space $\mathscr{D}^q(W_1, O)$ can be identified with q-forms f on $\tilde{W}$ with values in R^n such that $f(g_1x) = O(g_1)f(x)$, $g_1 \in G_1$. Similarly, using the equivalence T, the space $\mathscr{D}^q(W_2, U^O)$ can be identified with q-forms $\tilde{f}$ on $\tilde{W}$ with values in R^{rn} satisfying $\tilde{f}(gx) = \rho(g)\tilde{f}(x)$. For a form f on $\tilde{W}$ with values in R^n, let Sf be the form on $\tilde{W}$ with values in R^{rn} defined by

$$Sf(x) = \sum_{k=1}^{r} \oplus f(\alpha_k x).$$

If f satisfies $f(g_1x) = O(g_1)f(x)$, $g_1 \in G_1$, then

$$\begin{aligned} Sf(gx) &= \sum_{k=1}^{r} \oplus f(\alpha_k g x) \\ &= \sum_{k=1}^{r} \oplus \sum_{j=1}^{r} \dot{O}(\alpha_k g \alpha_j^{-1}) f(\alpha_j x) \\ &= \rho(g)(Sf)(x). \end{aligned}$$

Thus S defines a map of $\mathscr{D}^q(W_1, O)$ into $\mathscr{D}^q(W_2, U^O)$. The map is certainly injective. For $\tilde{f} = \sum \oplus f_j$ in $\mathscr{D}^q(W_2, U^O)$, an easy computation shows that f_1 is in $\mathscr{D}^q(W_1, O)$ and $\tilde{f} = Sf_1$, so S is also surjective. Finally, the Laplacian commutes with S, since it commutes with the action of G and acts componentwise.

Remark. For $\lambda \neq 0$, let $\tilde{\mathscr{E}}_q(\lambda)$ be the space of q-forms $\tilde{f}$ on $\tilde{W}$ satisfying $\Delta\tilde{f} = \lambda\tilde{f}$. The action of $G_2 = \pi_1(W_2)$ on W, commuting with the Laplacian, determines a representation U of G_2 as unitary operators on $\tilde{\mathscr{E}}_q(\lambda)$. The multiplicity $N_q(\lambda, O)$ of λ in $\mathscr{D}^q(W_1, O)$ is given by the intertwining number $I(O, U_{G_1})$ [6] of the representations O and U_{G_1}, the restriction of U to G_1. Thus Theorem 2.6 is simply a restatement of the Frobenius reciprocity theorem $I(U^O, U) = I(O, U_{G_1})$.

Besides the three preceding theorems, there is a very important property of R-torsion which we can state for the analytic torsion, but have not been able to prove. Namely, suppose W is a compact oriented manifold with boundary M. Let K be a triangulation of W, and L a subcomplex of K which is a triangulation of M. Let O be a representation of $\pi_1(K)$ by orthogonal matrices, and suppose that the homology groups of the corresponding complexes $C(K, O)$, $C(L, O)$, and $C(K/L, O)$ are free modules with preferred bases. Then [10, Theorem 3.2]

$$\log \tau_K(O) = \log \tau_L(O) + \log \tau_{K/L}(O) + \log \tau_H(O),$$

where τ_H is the torsion of the homology exact sequence of the triple, thought of as an acyclic chain complex of dimension $3N + 2$. The analog of this statement is formed by replacing τ_K, τ_L and $\tau_{K/L}$ by the analytic torsions of the Laplacians, respectively, on W with absolute boundary conditions, on M, and on W with relative boundary conditions. (The boundary conditions are described in Section 3.) If this property were proved for the analytic torsion, one could hope to prove the two torsions equal by copying the combinatorial invariance theorem [10, Theorem 7.1] for R-torsion, or Theorem 9.3 of [10].

The last-mentioned theorem establishes a construction of R-torsion (or more generally, Whitehead torsion) for manifolds in terms of self-indexing functions. This construction has led us to what seems to be a feasible program of proving that $T_W(O_1)/\tau_W(O_1) = T_W(O_2)/\tau_W(O_2)$ for two representations O_1 and O_2 of $\pi_1(W)$.

That is, let ϕ be a self-indexing function on W, and let $W_u = \phi^{-1}([0, u])$. If u is not an integer, W_u is a compact manifold with boundary $M_u = \phi^{-1}(u)$, and for a representation O of $\pi_1(W)$ we have the restriction of the corresponding vector bundle $E(O)$ to W_u. In Section 3 we will present definitions of R-torsion and analytic torsion for this situation, where the homology may not be trivial. In Section 7 we will prove that for the extended torsions, $\log(T_W(O_1)/T_W(O_2)) - \log(\tau_W(O_1)/\tau_W(O_2))$ is invariant under a change of metric on W which preserves the normal direction to the boundary.

But for any metric on W, the dual to $d\phi$ is a vector field X orthogonal to M_u for all u. Whenever $W_{u'} - W_u$ contains no critical points, the vector field X determines a diffeomorphism $F : W_u \to W_{u'}$. Since F preserves the normal vector X to the boundaries, we conclude by Theorem 7.1 that $\log(T_{W_u}(O_1)/T_{W_u}(O_2)) - \log(\tau_{W_u}(O_1)/\tau_{W_u}(O_2))$ is independent of u as long as one does not pass a critical point. For small u, both terms are trivially zero since W_u is a cell. If one could prove that both terms have the same jump as a critical point is crossed, then the equality of the two terms for W would follow.

As a final bit of evidence that T_W and τ_W may be equal, we point out that for lens spaces, the analytic torsion can be calculated explicitly and agrees with the R-torsion [14].

3. A Torsion for Riemannian Manifolds

The setting for the remainder of the paper will be essentially that of [10, Section 9]. W is a compact, oriented, C^∞ manifold of dimension N, whose boundary is the union of two disjoint, closed submanifolds M_1 and M_2. We do not exclude the possibilities that M_1, M_2, or both, are empty. O is a representation of the fundamental group $\pi_1(W)$ by orthogonal $n \times n$ matrices.

The relation between this situation and that described at the end of Section 2 is as follows. Let W' be a closed oriented manifold (the case of Section 2), or let W' be an h-cobordism: a compact oriented manifold with boundary the union of two disjoint closed submanifolds M_1' and M_2', each of which is a deformation retract of W'. Let ϕ be a self-indexing function on W' such that $\phi^{-1}(-1/2) = M_1'$. If u is not an integer, then $W = (W')_u = \phi^{-1}([-1/2, u])$ is a compact manifold whose boundary is the disjoint union of $M_1 = M_1'$ and $M_2 = \phi^{-1}(u)$. (M_1 is empty, of course, if W' is closed.)

In either case, let O' be a representation of the fundamental group of W' by orthogonal matrices. Since W is a submanifold of W', there is a natural homomorphism of $\pi_1(W)$ into $\pi_1(W')$, which composed with O' defines a representation O of $\pi_1(W)$. Thus we have the situation described in the first paragraph. We will eventually have to identify the bundle $E(O)$ on W with the restriction of $E(O')$ to W. But this is not difficult; the details are given in the proof of Corollary 8.

Let K be the simplicial complex of a C^m, $m > 2$, triangulation of W, which contains subcomplexes L_1 and L_2 triangulating M_1 and M_2. Let $C(\tilde{K}, \tilde{L}_1)$ be the relative chain group of the simply connected covering space $\tilde{K}$ of K, modulo that of L_1. As in Section 1, we may think of $C(\tilde{K}, \tilde{L}_1)$ as a module over the real group algebra $R(\pi_1)$. A preferred basis is given by choosing cells covering those in $K - L_1$. We define the chain complex $C(K, L_1; O)$ by

$$C(K, L_1; O) = R^n \underset{R(\pi_1)}{\otimes} C(\tilde{K}, \tilde{L}_1)$$

where the representation O is used to make R^n a right $R(\pi_1)$ module.

If the chain complex $C(K, L_1; O)$ happens to be acyclic, we can define the R-torsion of K modulo L_1 as in Section 1 or [10, Section 9]. The remarks following Definition 1.3 are valid; in particular, the R-torsion is invariant under subdivision and hence is a function of the pair (W, M_1). In the program described at the end of Section 2, however, the complex $C(K, L_1; O)$ will in general have nontrivial homology. We can then define the R-torsion only if we can choose a preferred basis of the homology classes.

In order to do this, we suppose that W is equipped with a Riemannian metric. The euclidean structure which the metric determines on the tangent space allows us to define a normal vector to the boundary at each boundary point of W. Accordingly, at a boundary point of W we can decompose a real differential form f into its normal and tangential components: $f = f_{\tan} + f_{\text{norm}}$. To be explicit, let η be the inward-pointing unit normal in the cotangent space at a boundary point. If f is a 1-form, then f_{norm} is the orthogonal projection of f on the subspace spanned by η. In general, we can write

$$f_{\text{norm}} = g \wedge \eta, \qquad \text{where} \quad {*g} = ({*f}) \wedge \eta. \tag{3.1}$$

The decomposition $f = f_{\tan} + f_{\text{norm}}$ is likewise defined, componentwise, for a differential form f with values in the vector bundle $E(O)$ associated with the representation O.

DEFINITION 3.2. A differential form f is said to satisfy relative boundary conditions at a boundary point of W if $f_{\tan} = (\delta f)_{\tan} = 0$ there. It satisfies absolute boundary conditions if $f_{\text{norm}} = (df)_{\text{norm}} = 0$.

We will denote by $\mathscr{D} = \mathscr{D}(W, O) = \sum \mathscr{D}^q(W, O)$ the space of C^∞ differential forms on W with values in $E(O)$ which satisfy relative boundary conditions at each point of the boundary M_1 and absolute boundary conditions at each point of M_2.

The boundary conditions introduced above are coercive for the Laplacian; in particular, the de Rham–Hodge theory holds [2]. That is, let $\mathscr{H} = \sum \mathscr{H}^q$ be the space of harmonic forms in $\mathscr{D}$ ($h \in \mathscr{H}$ means $h \in \mathscr{D}$ and $dh = \delta h = 0$). Let $A^q : \mathscr{H}^q \to C^q(K, L_1 ; O)$ denote the de Rham map defined by

$$A^q h(\xi \otimes e) = \int_e (\xi,\ h), \tag{3.3}$$

where e is a q-simplex in K, $\xi \in R^n$, and (,) denotes inner product in R^n. Then A^q is a one-one map of $\mathscr{H}^q$ onto a linear space of cocycles representing $H^q(K, L_1 ; O)$. (See the remark after Proposition 4.2).

For the present, note that $A^q h$ is indeed an element of $C^q(K, L_1 ; O)$. For if e is in L_1, $A^q h(\xi \otimes e) = 0$ since $h_{\tan} = 0$ on M_1. And if $\tilde{h}$ is the form obtained by lifting h to the simply connected covering space $\tilde{W}$, then for g in $\pi_1(W)$ acting as a deck transformation on $\tilde{W}$,

$$\begin{aligned} \int_{ge} (\xi, \tilde{h}) &= \int_e (\xi, \tilde{h} \circ g) \\ &= \int_e (\xi, O(g)\, \tilde{h}) \\ &= \int_e (\xi O(g), \tilde{h}). \end{aligned}$$

Now the Riemannian metric picks out a preferred base of $\mathscr{H}^q$ for us, viz., an orthonormal base. And we can use duality of forms, the de Rham map, and Poincaré duality to map $\mathscr{H}^q$ onto a set of representatives of homology classes in $C_q(K, L_1 ; O)$. Thus we obtain, as the image of an orthonormal base in $\mathscr{H}^q$, a preferred base of the homology classes by which to define the R-torsion.

Since we will make extensive use of the map described above, we will give a detailed definition. Let us start by reviewing the Poincaré duality for the complex K. Let K' be the barycentric subdivision of K, considered as another C^m triangulation of W. As an abstract simplicial complex, the vertices of K' are the simplexes of K. These are partially ordered by incidence in K, and a simplex in K' consists of a linearly ordered subset. K' becomes a triangulation of W by means of the barycentric coordinates in K.

For each q-simplex e of K, the dual $(N - q)$-cell $*e$ of e is the union of those simplexes in K' whose lowest vertex is e. Strictly speaking, $*e$ is not a C^m cell in W, but rather the union of C^m cells; but this will not affect the forthcoming definitions, since integration of an $(N - q)$-form over $*e$ is well defined. If e is in the subcomplex L_j, $j = 1, 2$, then e has, besides $*e$, another dual cell $*_j e$ of dimension $N - q - 1$, its dual in the submanifold M_j.

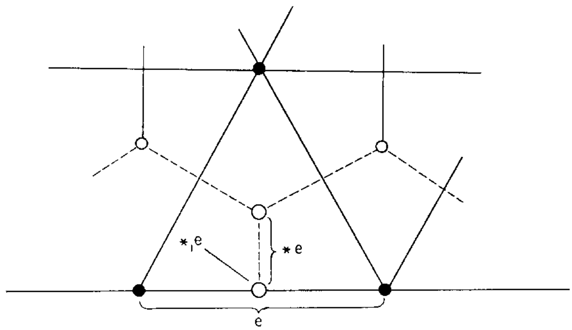

FIG. 1. A 1-simplex e in L_1 and its dual cells $*e$ of dimension 1 and $*^1 e$ of dimension 0.

The cells $*e$, for e in $K - L_1$, are just those which are disjoint from M_1 and meet M_2 only along their boundaries. These cells, together with the cells $*^2 e$, e in L_2, thus form a cell complex K^* in W.

If $L_2{}^*$ denotes the subcomplex of K^* consisting of cells $*^2 e$ for e in L_2, then the collection $*e$, e in $K - L_1$, is a base for the relative chain complex $C(K^*, L_2{}^*)$. The one-one correspondence $e \leftrightarrow *e$ of bases determines an isomorphism ρ_q of the chain group $C_q(K, L_1)$ onto the dual of $C_{N-q}(K^*, L_2{}^*)$. Using the orientation of K^* determined by the fundamental cycle in $C_N(K, L_1 \cup L_2)$, we have for c in $C_{q+1}(K, L_1)$, c^* in $C_{N-q}(K^*, L_2{}^*)$,

$$\langle \partial c^*, \rho_{q+1} c \rangle = (-1)^{N-q+1} \langle c^*, \rho_q \partial c \rangle. \tag{3.4}$$

(We write ∂ for the generic boundary operator in each chain group.) Lifting to the covering manifold $\tilde{W}$, we denote also by ρ_q the ensuing isomorphism of $C_q(K, L_1\,; O)$ onto $C^{N-q}(K^*, L_2{}^*; O)$.

Now define A_q to be the map $A_q = (-1)^{(N-1)q} \rho_q^{-1} A^{N-q} *$ of $\mathscr{H}^q$ into $C_q(K, L_1\,; O)$. That is, for h in $\mathscr{H}^q$, $*h$ is a harmonic $(N - q)$-form with values in $E(O)$, which satisfies by (3.1) the relative boundary conditions on M_1. We retain the symbol A^{N-q} for the de Rham map (3.3) of forms into $C^{N-q}(K^*, L_2^*; O)$. The composition A_q is given explicitly by the formula

$$A_q h = (-1)^{(N-1)q} \sum \int_{*e} (\xi, *h) \cdot (\xi \otimes e) \tag{3.5}$$

where the summation runs over the q-simplexes e of $K - L_1$ and the elements ξ of an orthonormal base of R^n. Stokes theorem, the definition of ∂, and (3.4) show directly that

$$A_q(\delta h) = \partial A_q h = 0.$$

Finally, the de Rham–Hodge theory shows that A_q maps $\mathscr{H}^q$ onto a set of representatives of the homology classes in $C_q(K, L_1; O)$.

Definition 3.6. For the situation described in this section, the R-torsion $\tau_{K,L_1}(O)$ of the pair (K, L_1) will be that given by Definition 1.1 for the chain complex $C(K, L_1; O)$, where a preferred base of the homology classes is chosen as the image under A_q of an orthonormal base of $\mathscr{H}^q$, for each q.

Remark. Since a change of orthonormal base in $\mathscr{H}^q$ is given by an orthogonal matrix, the R-torsion defined above is independent of the choice of one orthonormal base. It is again independent of the choice of representatives in $\tilde{K}$ of the simplexes of K (see the remark after Definition 1.3). Hence, the R-torsion τ_{K,L_1} is a function only of the representation O, the triangulation (K, L_1) and the Riemannian metric assigned to W.

Proposition 3.7. *The torsion* τ_{K,L_1} *of Definition* 3.6 *does not depend on the choice of the triangulation of* W.

The proof will be given at the end of Section 4. In Section 7 we will describe the behavior of the R-torsion of Definition 3.6 resulting from certain variations of the Riemannian metric on W.

4. The de Rham Theorem for a Compact Manifold with Boundary

The proof of Proposition 3.7 depends on a form of de Rham's theorem which relates the duality between homology and cohomology with the duality of differential forms on a Riemannian manifold. It was proved by Kodaira [4] for closed manifolds, and no important changes are needed to apply his proof to manifolds with boundary.

Let us start by summarizing the setting for the theorem, as described in Section 3. W is a C^∞ compact oriented Riemannian manifold of dimension N, with boundary consisting of disjoint closed submanifolds M_1 and M_2, either or both of which may be empty. O is a representation of the fundamental group $\pi_1(W)$ by orthogonal $n \times n$ matrices, and $E(O)$ is the associated vector bundle.

K is a C^m triangulation of W, $m \geqslant 2$, with subcomplexes L_1 and L_2 triangulating the boundary manifolds M_1 and M_2. $C(K, L_1; O)$ is the chain complex of the relative complex K modulo L_1 formed by the action of $\pi_1(W)$ on the covering space K and by the representation O. K^* is the dual cell complex formed from the barycentric subdivision of K and $C(K^*, L_2^*; O)$ the associated chain complex of K^* modulo L_2^* Finally, the isomorphism $\rho_q : C_q(K, L_1; O) \to C^{N-q}(K^*, L_2^*; O)$ is defined by the pairing of the base elements e, $*e$, where e is in $K - L_1$ and $*e$ is its dual cell in W.

Rather than consider the space $\mathscr{D}$ of Definition 3.2 on the subspace $\mathscr{H}$ of harmonic forms in $\mathscr{D}$, we will now define two spaces of forms, whose intersection is $\mathscr{H}$.

Definition 4.1. Let $\mathscr{Z}_q$ be the space of C^∞ differential q-forms f on W with values in $E(O)$, satisfying

$$\delta f = 0,$$
$$f_{\text{norm}} = 0 \qquad \text{on } M_2 .$$

Let $\mathscr{Z}^q$ be the space of C^∞ q-forms f with values in $E(O)$ satisfying

$$df = 0,$$
$$f_{\tan} = 0 \qquad \text{on } M_1 .$$

Proposition 4.2 (de Rham's theorem). *The map A^q defined by (3.3) carries $\mathscr{Z}^q$ onto the space of cocycles in $C^q(K, L_1; O)$, and $A_q = (-1)^{(N-1)q}\rho_q^{-1}A^{N-q} *$ (see (3.5)) carries $\mathscr{Z}_q$ onto the space of cycles in $C_q(K, L_1; O)$. If f is in $\mathscr{Z}_q$ and g is in $\mathscr{Z}^q$, then*

$$\begin{aligned}(f, g) &= \int_W f \wedge *g \\ &= (-1)^{(N-1)q} \sum_{K-L_1} \left(\int_{*e} *f, \int_e g\right) \\ &= (A_q(f), A^q(g)). \end{aligned} \tag{4.3}$$

Remark. By (4.3), $A_q(f)$ is a boundary in $C_q(K, L_1\,; O)$ if and only if f is orthogonal to $\mathscr{Z}^q$. Hence as a corollary to 4.2, A_q maps $\mathscr{H}^q = \mathscr{Z}_q \wedge \mathscr{Z}^q$ one-one onto a set of representatives of the homology classes in $C_q(K, L_1\,; O)$.

Kodaira's proof for closed manifolds proceeds by using a "Poincaré lemma" to pull f back to a neighborhood of the q-skeleton of K and g to a neighborhood of the $(N - q)$-skeleton of K^*. Once this is done, the supports of f and g will intersect only in disjoint neighborhoods of the barycenters of q-simplexes of K, and the proof of (4.3) becomes a local affair.

One way to apply this procedure to our situation is to replace f by a form vanishing near M_2 and g by a form vanishing near M_1. We will do this, and show that the forms will continue to vanish in these regions during the pulling back described above, in a pair of lemmas. Then we will be able to apply Kodaira's proof with only minor changes.

LEMMA 4.4. *Suppose f is in $\mathscr{Z}_q$ and g in $\mathscr{Z}^q$. Then there are forms f_N in $\mathscr{Z}_q$ and g_0 in $\mathscr{Z}^q$ such that the support of f_N is disjoint from M_2, the support of g_0 is disjoint from M_1, and*

$$(f, g) = (f_N\,, g_0),$$
$$(A_q(f), A^q(g)) = (A_q(f_N), A^q(g_0)).$$

Proof. If $q = 0$, $\mathscr{Z}^q$ contains only the null function unless M_1 is empty, so $g_0 = g$ will trivially have support disjoint from M_1. So suppose $q > 0$.

Let x^N be a smooth function without critical points from a neighborhood $N(M_1)$ of M_1 onto $[0, 1)$ such that $M_1 = (x^N)^{-1}(0)$. Such a function can easily be constructed by using a partition of unity. Let X be the vector field in this neighborhood which is dual to dx^N, and integrate along X as in Section 3 of [9] to construct a diffeomorphism of $M_1 \times (0, 1)$ onto $N(M_1)$. If $x = (x^1,..., x^{N-1})$ is a local chart in M_1 and $0 \leqslant x^N < 1$, then under the diffeomorphism (x, x^N) becomes a chart in $N(M_1)$. Because we have chosen X dual to dx^N, this chart has the property that $(dx^N)_{\tan} = 0$, while $(dx^i)_{\text{norm}} = 0$ for $i < N$, on M_1.

Writing a form g in $N(M_1)$ as

$$g = \sum_{i_1 \cdots i_q} g_{i_1 \cdots i_q}\, dx^{i_1} \wedge \cdots \wedge dx^{i_q},$$

where $g_{i_1 \ldots i_q}$ is an alternating function with values in R^n, set

$$R_1 g = \sum_{i_1 \cdots i_{q-1}} (R_1 g)_{i_1 \cdots i_{q-1}} \, dx^{i_1} \wedge \cdots \wedge dx^{i_{q-1}},$$

where

$$(R_1 g)_{i_1 \cdots i_{q-1}}(x, x^N) = (-1)^{q+1} \int_0^1 g_{i_1 \cdots i_{q-1}, N}(x, tx^N) \, x^N \, dt.$$

Since this formula is invariant under change of local coordinates in M_1, it serves to define $R_1 g$ consistently throughout $N(M_1)$. Note that $R_1 g$ vanishes for $x^N = 0$. Computation shows that if

$$\begin{aligned} g_{\tan}(x) &= \sum_{i_1 \cdots i_q < N} g_{i_1 \cdots i_q}(x, 0) \, dx^{i_1} \wedge \cdots \wedge dx^{i_q} \\ &= 0, \qquad x \in M_1, \end{aligned}$$

then

$$dR_1 g + R_1 \, dg = g. \tag{4.5}$$

Now for g in $\mathscr{L}^q$, apply R_1 to the restriction of g to $N(M_1)$, and let

$$g_0 = g - d(\psi R_1 g),$$

where ψ is a smooth function vanishing outside $N(M_1)$ and equal to one in the neighborhood $0 \leqslant x^N \leqslant 1/2$ of M_1. (4.5) shows that g_0 vanishes for $x^N \leqslant 1/2$ and so is in $\mathscr{L}^q$.

If f is in $\mathscr{L}_q$, then

$$\begin{aligned} (A_q(f), A^q(g)) - (A_q(f), A^q(g_0)) &= (A_q(f), A^q(d\psi R_1 g)) \\ &= (A_q(f), \partial^* A^q(\psi R_1 g)) \\ &= (\partial A_q(f), A^q(\psi R_1 g)) \\ &= (A_q(\delta f), A^q(\psi R_1 g)) \\ &= 0. \end{aligned}$$

On the other hand, applying Stokes' theorem to

$$d(\psi R_1 g \wedge {*f}) = (g - g_0) \wedge {*f},$$

the resulting integrals over M_1 and M_2 vanish since $({*f})_{\tan} = f_{\text{norm}} = 0$ on M_2 and since $R_1 g = 0$ on M_1. So

$$(f, g) = (f, g_0).$$

We construct f_N (for $g < N$) by applying a similar operation in a neighborhood of M_2 to the closed form $*f$, and this completes the proof of 4.4.

LEMMA 4.6. *Let e be an r-simplex in $K - (L_1 \cup L_2)$, and suppose that f is a C^∞ q-form whose support is disjoint from M_2 and is contained in a neighborhood of e. Suppose that the support of δf is contained in a neighborhood of ∂e. Then if $r > q$, there is a C^∞ $(q+1)$-form $R_e f$ such that the support of $f - \delta R_e f$ is disjoint from M_2 and is contained in a neighborhood of ∂e.*

Let e be an r-simplex in K, and suppose that g is a C^∞ q-form whose support is disjoint from M_1 and is contained in a neighborhood of the dual $(N - r)$-cell $*e$. Suppose that the support of dg is contained in a neighborhood of $\partial(*e)$. Then if $r < q$, there is a C^∞ $(q - 1)$ form $R_{*e}\, g$ such that the support of $g - dR_{*e}\, g$ is disjoint from M_1 and is contained in a neighborhood of $\partial(*e)$.

Proof. Given the r-simplex e in $K - (L_1 \cup L_2)$, there is a C^∞ chart $x^1, \ldots, x^N$ defined in a neighborhood of e such that, given $\delta > 0$,

$$e \subset \left\{ x : \sum_{r+1}^{N} (x^i)^2 < \delta, \sum_{1}^{r} (x^i)^2 < 1 \right\},$$

$$\partial e - M_2 \subset \left\{ x : 1 - \delta < \sum_{1}^{r} (x^i)^2 < 1 \right\},$$

while if e has a face in M_2, $x^1 = 0$ on M_2 and $x^1 > 0$ on $e - M_2$. We can assume that

$$f = 0 \qquad \text{if} \quad \sum_{r+1}^{N} (x^i)^2 \geqslant \delta \quad \text{or} \quad \sum_{1}^{r} (x^i)^2 \geqslant 1,$$

$$\delta f = 0 \qquad \text{if} \quad \sum_{1}^{r} (x^i)^2 \leqslant 1 - \delta,$$

and $f = 0$ for $0 \leqslant x \leqslant \delta$ if e has a face in M_2.

Writing the closed $(N - q)$-form $*f$ as

$$*f = \sum_{i_1 \cdots i_{N-q}} (*f)_{i_1 \cdots i_{N-q}} dx^{i_1} \wedge \cdots \wedge dx^{i_{N-q}},$$

set

$$Q_e *f = \sum_{i_1 \cdots i_{N-q-1}} (R_e *f)_{i_1 \cdots i_{N-q-1}} dx^{i_1} \wedge \cdots \wedge x^{i_{N-q-1}},$$

with

$$(Q_e *f)_{i_1 \cdots i_{N-q-1}}(x) = \sum_{i=1}^{r} x^i \int_0^1 t^{\#} (*f)_{i, i_1 \cdots i_{N-q-1}} (tx^1, ..., tx^r, x^{r+1}, ..., x^N)\, dt,$$

where $\#$ is the number of integers among $i_1 \cdots i_{N-q-1}$ which do not exceed r.

$Q_e *f$ is defined in the given neighborhood of e, and vanishes if $\sum_{r+1}^{N} (x^i)^2 \geqslant \delta$. $Q_e *f$ vanishes also for $0 \leqslant x^1 \leqslant \delta$ if e has a face in M_2. $Q_e * \delta f = (-1)^q Q_e d *f$ vanishes in the same regions, of course, and vanishes also if $\sum_1^N (x^i)^2 \leqslant 1 - \delta$. Note that in the integral defining the components of $Q_e * \delta f$, we will have $\# > 0$, since $r > q$. Using this fact, computation shows that

$$dQ_e *f + Q_e d *f = f \tag{4.7}$$

in the neighborhood.

Now let ψ be a C^∞ function of $x^1, ..., x^r$ such that

$$\psi = 0 \quad \text{if} \quad \sum_1^r (x^i)^2 \geqslant 1,$$

$$\psi = 1 \quad \text{if} \quad \sum_i^r (x^i)^2 \leqslant 1 - \delta,$$

and define

$$R_e f = (-1)^q * \psi Q_e *f,$$

extending $R_e f$ to be a C^∞ $(q + 1)$-form vanishing outside the given neighborhood of e. Then (4.7) shows that $R_e f$ has the properties described in 4.6.

The function $R_{*e}\, g$ is again constructed by the dual process applied in a neighborhood of the cell $*e$.

LEMMA 4.8. *Let U be a neighborhood in W which is homeomorphic to the open unit cube in R^N.*

Suppose f is a C^∞ q-form, $q < N$, with support contained in U, and suppose $\delta f = 0$ if $1 \leqslant q < N$, $\int {}^*f = 0$ if $q = 0$. Then there is a C^∞ $(q+1)$-form f' with support contained in U such that $f = \delta f'$.

Suppose g is a C^∞ q-form, $q > 0$, with support contained in U, and suppose $df = 0$ if $0 < q < N$, $\int g = 0$ if $q = N$. Then there is a C^∞ $(q-1)$-form g' with support contained in U such that $g = dg'$.

This is precisely Lemma 3.1 of [4], and we will not repeat the proof.

LEMMA 4.9. *Let U be a neighborhood in W which is homeomorphic to the closed unit cube in R^N, and suppose the face $(x^N)^{-1}(1)$ lies in M_1 or M_2.*

Suppose f is a C^∞ q-form, $q < N$, with support contained in U, and suppose $\delta f = 0$. Then there is a C^∞ $(q+1)$-form with support contained in U such that $f = \delta f'$.

Suppose g is a C^∞ q-form, $q > 0$, with support contained in U, and suppose $dg = 0$. Then there is a C^∞ $(q-1)$-form g' with support contained in U such that $g = dg'$.

Proof. The proof uses the same operator R_1 as in the proof of Lemma 4.4, but applied in the opposite direction, so to speak. That is, writing

$$g = \sum_{i_1 \cdots i_q} g_{i_1 \cdots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q},$$

set

$$R_1 g = \sum_{i_1 \cdots i_{q-1}} (R_1 g)_{i_1 \cdots i_{q-1}} dx^{i_1} \wedge \cdots \wedge dx^{i_{q-1}},$$

where

$$(R_1 g)_{i_1 \cdots i_{q-1}}(x) = (-1)^{q+1} \int_0^1 g_{i_1 \cdots i_{q-1}, N}(x', \ldots, x^{N-1}, tx^N)\, x^N\, dt.$$

Since the face $(x^N)^{-1}(1)$ lies in the boundary of W, $g' = R_1 g$ is a C^∞ form with support in U, without further alteration. And by (4.5), we have $g = dg'$ when $dg = 0$.

We will now proceed to the proof of (4.3). Let K_q be the union of the simplexes of K of dimension $\leqslant q$, and let K^*_{N-q} be the union of the dual cells $^*e$, $e \in K$ of dimension $\leqslant N - q$. Given f in $\mathscr{X}_q$ and g in $\mathscr{X}^q$, we construct q-forms f_q and g_{N-q} such that (compare with Lemma 3.2 of [4]):

The support of f_q is disjoint from M_2 and is contained in a given neighborhood of $M_1 \cup K_q$;

The support of g_{N-q} is disjoint from M_1 and is contained in a given neighborhood of $M_2 \cup K^*_{N-q}$;

$$\delta f_q = dg_{N-q} = 0;$$
$$(f_q, g_{N-q}) = (f, g);$$
$$(A_q(f_q), A^q(g_{N-q}) = (A_q(f), A^q(g)).$$

We construct f_q by defining inductively coclosed q-forms f_r, $q \leqslant r \leqslant N$, such that the support of f_r is disjoint from M_2 and is contained in a given neighborhood of $M_1 \cup K_r$, and such that

$$(f_r, g) = (f, g), \tag{4.10}$$
$$(A_q(f_r), A^q(g) = (A_q(f), A^q(g)), \tag{4.11}$$

for g in $\mathscr{X}^q$.

Note that to start the procedure we have the form f_N given by Lemma 4.4. So suppose that $r > q$, and that f_r has been defined. For each r-simplex e in K, let ψ_e be a nonnegative function vanishing outside a neighborhood of e, such that $\{\psi_e, e \in K_r\}$ form a partition of unity on K_r. We can suppose that the q-form $(1 - \psi_e) f_r$ has support in a given neighborhood of ∂e, and $\psi_e f_r = 0$ for $e \in L_2$. For e not in $L_1 \cup L_2$, $\psi_e f_r$ satisfies the hypothesis of Lemma 4.6. Set

$$f_{r-1} = f_r - \sum_{e \in K-(L_1 \cup L_2)} \delta R_e(\psi_e f_r).$$

Since $f_r = \sum \psi_e f_r$, application of 4.6 to each form $\psi_e f_r - \delta R_e(\psi_e f_r)$ shows that f_{r-1} has support disjoint from M_2 and contained in a given neighborhood of $M_1 \cup K_r$. (4.10) and (4.11) follow just as in the proof of Lemma 4.4.

Again, g_{N-q} is constructed by the dual procedure.

We can assume that the neighborhoods of $M_1 \cup K_q$ and $M_2 \cup K^*_{N-q}$ have been chosen so that the support of $g_{N-q} \wedge {*f_q}$ is contained in the union of disjoint neighborhoods of the barycenters of q-simplexes of $K - (L_1 \cup L_2)$. Thus if we suitably define smooth functions $\{\psi_e, e \in K_q\}$

and $\{\psi_{*e}, e \in K^*_{N-q}\}$ whose restrictions to K_q and K^*_{N-q}, respectively, form partitions of unity, then

$$\begin{aligned}(f, g) &= (f_q, g_{N-q}) \\ &= \sum_{e \in K-(L_1 \cup L_2)} (\psi_e f_q, \psi_{*e} g_{N-q}),\end{aligned}$$

$$\begin{aligned}(A_q(f), A^q(g)) &= (A_q(f_q), A^q(g_{N-q})) \\ &= \sum_{e \in K-(L_1 \cup L_2)} (A_q(\psi_e f_q), A^q(\psi_{*e} g_{N-q})).\end{aligned}$$

Hence it suffices to prove (4.3) for each of the summands above.

In other words, it suffices to prove (4.3) under the assumption that f has support in a neighborhood of a q-simplex e of $K - (L_1 \cup L_2)$, g has support in a neighborhood of $*e$, and δf and dg have support in neighborhood of ∂e and $\partial(xe)$, respectively. But this is just Lemma 3.3 of [4]; for clarity (and since a minor change is necessary), we will repeat Kodaira's proof.

For $q = 0$, N, the proof is trivial since (if $q = 0$) the function g is constant in the support of f. The proof for $0 < g < N$ proceeds by induction. Let e' be a $(q + 1)$ simplex such that e lies in $\partial e'$. Applying Lemma 4.8, or Lemma 4.9 if e' meets L_1, there are forms f', f'' with supports in neighborhoods of $\partial e' - e$ and e', respectively, such that

$$\begin{aligned}\delta f' &= \delta f, \\ \delta f'' &= f - f'.\end{aligned}$$

We can assume that f' vanishes on the support of g, so that

$$\begin{aligned}(f, g) &= (\delta f'', g) \\ &= (f'', dg).\end{aligned}$$

But there is a $(q + 1)$-form g'' with support in a neighborhood of $*e'$, such that $g'' = dg$ near e', $dg' = 0$ outside a neighborhood of $\partial(*e')$. By the induction hypothesis, (4.3) holds for the forms f'', g''. Then

$$\begin{aligned}(f, g) &= (f'', g'') \\ &= (A_q(f''), A^q(g'')) \\ &= (-1)^{(N-1)(q+1)} \left(\int_{*e'} *f'', \int_{e'} g''\right) \\ &= (-1)^{(N-1)q} \left(\int_{*e} *f, \int_e g\right),\end{aligned}$$

proving the induction step.

To see that A_q maps $\mathscr{L}_q$ onto the space of cycles in $C_q(K, L_1 ; O)$, Kodaira's proof can again be used. One constructs, by induction, linear maps B_q of $C_q(K, L_1 ; O)$ into C^∞ q-forms on W with the properties:

For each c in C_q, $B_q(c)$ vanishes in a neighborhood of M_2;

For each q-simplex e of $K - L_1$, $B_q(e)$ has support in a neighborhood of e;

$$A_q B_q(c) = c;$$

$$B_q(\partial c) = \delta B_q(c).$$

It does not seem worthwhile to repeat the details in this case; we refer the reader to [4]. We will point out, however, that in defining $B_q(e)$ when e meets L_1, one must use Lemma 4.9 rather than 4.8. The reason for this is clear when $q = 1$. $B_1(e)$ is to be defined as a solution of $\delta B_1(e) = B_0(\partial e)$. Lemma 4.8 cannot be applied if e meets L_1, since then $\int {}^*B_0(\partial e) = 1$.

Of course, remarks of a similar nature apply in constructing dual maps B^q to show that A^q maps $\mathscr{L}^q$ onto the space of cocycles.

We turn to the proof of Proposition 3.7. As in the proof of Lemma 9.1 of [10], we want to show that $\tau_{K,L_1} = \tau_{K',L_1'}$, when K' is a triangulation of W which subdivides K. To do this, however, we can apply the Combinatorial Invariance Theorem 7.1 of [10]. We need only check that the preferred bases of the homology groups $H_q(K, L_1 ; O)$ and $H_q(K', L_1'; O)$ correspond under the subdivision operator.

That is, we have chosen a fixed orthonormal base (h_α) of the space $\mathscr{H}^q$ of harmonic forms in $\mathscr{D}^q(W, O)$, $0 \leqslant q \leqslant N$. The torsion τ_{K,L_1} was defined by using the preferred base of $H_q(K, L_1 ; O)$ represented by the cycles $A_q(h_\alpha)$, and $\tau_{K',L_1'}$ by the base of $H_q(K', L_1'; O)$ represented by the cycles $A_q'(h_\alpha)$.

We have also the subdivision operator S of $C_q(K, L_1 ; O)$ into $C_q(K', L_1'; O)$ given by

$$Se = \sum_{e' \subset e} e'.$$

As is well known, S determines an isomorphism of $H_q(K, L_1 ; O)$ onto $H_q(K', L_1'; O)$, and the Combinatorial Invariance Theorem states that the torsions of the two complexes are equal if the preferred bases of the homology groups correspond under this isomorphism.

So we have to show that the cycles $(A_q'(h_\alpha))$ and $(SA_q(h_\alpha))$ represent unitarily equivalent bases of $H_q(K', L_1'; O)$, which is not immediately obvious.

What is obvious is that for any q-form g,

$$\int_e g = \sum_{e' \subset e} \int_{e'} g.$$

But then for f in $\mathscr{L}_q$ and g in $\mathscr{L}^q$,

$$\begin{aligned}
(SA_q(f), A'^q(g)) &= (-1)^{(N-1)q} \sum_{e \in K - L_1} \sum_{e' \subset e} \left(\int_{*e} *f, \int_{e'} g \right) \\
&= (-1)^{(N-1)q} \sum_{e \in K - L_1} \left(\int_{*e} *f, \int_{e} g \right) \\
&= (A_q(f), A^q(g)) \\
&= (f, g) \\
&= (A_q'(f), A'^q(g)).
\end{aligned}$$

According to the remark after Proposition 4.2, this means that $SA_q(f)$ and $A_q'(f)$ differ by a boundary. In particular, the cycles $(A_q'(h_\alpha))$ and $(SA_q(h_\alpha))$ represent the same base of $H_q(K', L_1'; O)$, and so the Combinatorial Invariance Theorem can be applied to prove Proposition 3.7.

5. The Heat Kernel on W

In this section we will construct and derive some properties of the fundamental solution of the initial-value problem

$$\begin{aligned}
&\frac{\partial}{\partial t} f(x, t) = \Delta f(x, t), \qquad x \in W, \quad t > 0; \\
&\lim_{t \to 0} f(x, t) = f(x);
\end{aligned} \tag{5.1}$$

for forms in the space $\mathscr{D} = \mathscr{D}(W, O)$ of Definition 3.2.

The initial-value problem (5.1) for real forms on a manifold with boundary was investigated by Conner in [2]. He showed that in the Hilbert space of square summable forms, the Laplacian on forms satisfying either the relative or absolute boundary conditions of 3.2

extends to the generator of a semigroup of compact self-adjoint operators, each of which commutes with d and δ. Conner's results, of course, extend immediately to forms with values in the bundle $E(O)$.

The operators of the semigroup are given by a kernel, which is the fundamental solution of (5.1). We will construct the fundamental solution by the parametrix method of E. E. Levi; this will provide us with the local estimates needed in Sections 6 and 7. The parametrix method was used by Milgram and Rosenbloom [7] to construct the fundamental solution of the heat equation for forms on a closed manifold: the modification of the parametrix to make it satisfy the boundary conditions follows the integral equation method used, for instance, by Conner [2]. A very careful exposition of these methods is presented by Friedman [3], and we will refer to his work for a number of estimates and calculations. We depart, however, from these methods by using a specialized form of the parametrix, as in [11], which makes evident the property of the heat kernel used in the proof of Theorem 2.1.

In order to describe the local estimates of the heat kernel, it is useful to bring in a distance function ρ on W. This is a function of the pair of points (x, y) of W with the properties

$$\rho^2(x, y) \text{ is } C^\infty \text{ on } W \times W;$$

$$\rho(x, x) = 0, \qquad \rho(x, y) > 0 \qquad \text{for} \quad x \neq y;$$

$$\frac{\partial^2}{\partial x^i\, \partial y^j}\, \rho^2(x, y) = g_{ij}(x) \qquad \text{when} \quad y = x.$$

Such a function can easily be constructed using local coordinates and a partition of unity. Set

$$k(x, y, t) = Kt^{-N/2} e^{-c\rho^2(x,y)/t}; \tag{5.2}$$

K and c are generic constants, which may depend, for instance, on a choice of local coordinates.

Proposition 5.3. *Given a continuous form f on W as initial value, the unique solution in $\mathscr{D}^q(W, O)$ of* (5.1) *is given by*

$$f(x, t) = (P(t)f)(x)$$

$$= \int P(x, y, t) \wedge {}^*f(y),$$

where the kernel

$$P(x, y, t) = \sum P_{i_1\cdots i_q; j_1\cdots j_q}(x, y, t)\, dx^{i_1} \wedge \cdots \wedge dx^{i_q}; dy^{j_1} \wedge \cdots \wedge dy^{j_q}$$

is a symmetric double form which for each $t > 0$ belongs to $\mathscr{D}^q$ as a function of each variable on W.

The kernel P has the property

$$d_x P(x, y, t) = \delta_y P(x, y, t),$$

and satisfies the bounds

$$\left| \frac{\partial^{m+n}}{(\partial x^i)^m (\partial y^j)^n} P_{i_1\cdots i_q; j_1\cdots j_q}(x, y, t) \right| \leqslant t^{-(m+n)/2} k(x, y, t), \qquad m, n = 0, 1, \quad (5.4)$$

for $0 < t < t_0$.

Finally, if x is in the interior of W, the kernel has the local asymptotic expansion

$$P(x, x, t) = \sum_0^n t^{-(N-2m)/2} C_m(x) + o(t^{n-N/2}) \qquad (5.5)$$

as $t \to 0$, where each C_m is a smooth double form, and where the convergence of the remainder term is uniform on any compact subset of the interior of W.

Remark. We include the asymptotic expansion (5.5) in order to verify, for completeness sake, the claim made in the proof of Theorem 2.1 concerning the kernel K_s of $(-\Delta_q)^{-s}$. For suppose W is closed and suppose that the cohomology of W with coefficients in the bundle $E(O)$ is trivial. Then the expansion (5.5) holds at each point x of W, and, since Δ is strictly negative, $P(x, x, t)$ decreases exponentially as $t \to \infty$. Thus for N odd

$$\begin{aligned} K_s(x, x) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} P(x, x, t)\, dt \\ &= \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} P(x, x, t)\, dt \\ &\quad + \frac{1}{\Gamma(s)} \sum_{k<N/2} \frac{C_k(x)}{(s + k - N/2)} \\ &\quad + \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} R(x, t)\, dt, \end{aligned} \qquad (5.6)$$

where each component of $R(x, t)$ is bounded by a multiple of $t^{1/2}$. Each term on the right is meromorphic in the half-plane $\operatorname{Re} s > -1/2$ and vanishes at $s = 0$ because of the factor $1/\Gamma(s)$.

Before embarking on the construction of the heat kernel P, which is rather lengthy, we will state and prove a corollary of Proposition 5.3.

COROLLARY 5.7. *Let $\mathscr{H}^q$ be the space of harmonic forms in $\mathscr{D}^q(W, O)$; i.e., $h \in \mathscr{H}^q$ if and only if $h \in \mathscr{D}^q$ and $dh = \delta h = 0$.*

Let f be a C^∞ q-form on W which is orthogonal to $\mathscr{H}^q$. Then

$$Gf = \int_0^\infty P(t) f \, dt$$

is in $\mathscr{D}^q$ and satisfies

$$\Delta Gf = -f.$$

If f is C^∞ on W and satisfies $f_{\mathrm{tan}} = 0$ on M_1, $f_{\mathrm{norm}} = 0$ on M_2, then f has the Hodge decomposition

$$f = dg_1 + \delta g_2 + h,$$

where g_1 is in the space $\mathscr{L}_{q-1}$ of Definition 4.1, dg_1 is in $\mathscr{L}^q$, g_2 is in $\mathscr{L}^{q+1}$, and δg_2 is in $\mathscr{L}_q$.

If f is a square integrable q-form on W, and

$$(f, dg) = 0, \qquad g \in \mathscr{D}^{q-1}(W, O),$$
$$(f, \delta g) = 0, \qquad g \in \mathscr{D}^{q+1}(W, O),$$

then f is in $\mathscr{H}^q$.

Proof. The operator P is clearly compact on square integrable forms on W, and has the spectral representation

$$P(x, y, t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \varphi_n(x)\, \varphi_n(y),$$

where $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty$, and φ_n runs through an orthonormal base of eigenforms of Δ corresponding to the eigenvalue $-\lambda_n$. Since the eigenforms are complete, a square integrable form f which satisfies $(f, \varphi_n) = 0$ for $n > 0$ will be harmonic. But for $n > 0$ we can write

$$\varphi_n = + \frac{1}{\lambda_n} (d\delta\varphi_n + \delta d\varphi_n),$$

and the statement of the last paragraph of 5.7 follows immediately.

It also follows from the spectral representation that if f is orthogonal to $\mathscr{H}^q$, then the L^2 norm of $P(t)f$ decreases exponentially as $t \to \infty$. Since for each $t > 0$, the operator $P(t)$ is a contraction in the L^2 norm, the integral which defines Gf converges in the L^2 norm. In fact, setting

$$g_n = \int_{1/n}^{n} P(t) f \, dt,$$

g_n is in $\mathscr{D}^q$ for each n, and converges to Gf in L^2. Using the heat equation,

$$\begin{aligned} \Delta g_n &= P(n) f - P(1/n) f \\ &\to -f \end{aligned}$$

in L^2 as $n \to \infty$. In particular, Gf is a weak solution of $\Delta g = -f$:

$$(Gf, \Delta\varphi) = -(f, \varphi), \qquad \varphi \in \mathscr{D}^q.$$

Now Morrey [12, Lemma 4.5] has shown that the quadratic form $(df, df) + (\delta f, \delta f)$ is coercive for either of the boundary conditions $f_{\text{norm}} = 0$ or $f_{\tan} = 0$. This allows application of the theory of elliptic operators (see, for instance, [1], especially pp. 141–144) to show that the weak solution Gf is a strong solution and in fact belongs to $\mathscr{D}^q$. We will indicate the notation and some of the results of this theory as it applies to our situation. Some of this will in turn be used also in the proof of Proposition 6.4.

To begin with, let f, g be C^∞ q-forms on W with values in $E(O)$. For $\alpha = (\alpha_1, \ldots, \alpha_m)$ an m-tuple of integers chosen from $1, \ldots, N$, and for a local chart $x^1, \ldots, x^N$, define the q-form $(\partial/\partial x^\alpha) f$ by

$$\left(\frac{\partial}{\partial x^\alpha} f\right)_{i_1 \cdots i_q} = \frac{\partial^m}{(\partial x^{\alpha_1}) \cdots (\partial x^{\alpha_m})} (f_{i_1 \cdots i_q}).$$

For β another such m-tuple, set $g^{\alpha\beta} = \prod g^{\alpha_j \beta_j}$. The N-form

$$\sum g^{\alpha\beta} (\partial/\partial x^\alpha) f \wedge {}^* (\partial/\partial x^\beta) g$$

does not depend on the choice of coordinates; so we can define

$$(f, g)_m = \int_W \sum_{\alpha,\beta} g^{\alpha\beta} \frac{\partial}{\partial x^\alpha} f \wedge * \frac{\partial}{\partial x^\beta} g.$$

This is a scalar product in the space of C^∞ q-forms; denote by $H_m(W, O)$ the Hilbert space formed by completion in the norm

$$\| g \|_m = \{(g, g)_m + (g, g)_0\}^{1/2}.$$

The Sobolev inequality [1, p. 32] implies that if g is in $H_m(W, O)$, then f is C^l for $l < m - N/2$. For $l \leqslant m$, g has strong derivatives of order l, defined as L^2 limits. In particular, dg and δg are defined for g in $H_1(W, O)$.

Let $V = V(W, O)$ be the closed subspace of $H_1(W, O)$ spanned by the C^∞ q-forms g which satisfy $g_{\tan} = 0$ on M_1 and $g_{\text{norm}} = 0$ on M_2. There exist constants c_1, c_2 such that for g in V

$$(dg, dg) + (\delta g, \delta g) \geqslant c_1(g, g)_1 - c_2(g, g)_0 .$$

This inequality expresses the coerciveness of the quadratic form over V. The proof starts with the application of Green's theorem

$$(dg, dg) + (\delta g, \delta g) = -(g, \Delta g) + \int_{M_2} g \wedge *dg - \int_{M_1} \delta g \wedge *g.$$

The principal part of the Laplacian Δ is just the operator

$$\sum (\partial/\partial x^j)\, g^{jk}(\partial/\partial x^k)$$

in the notation we have been using. Thus we have

$$(g, \Delta g) + (g, g)_1 = \sum_{j,k} \int_W \frac{\partial}{\partial x^j} \left(g^{jk} g \wedge * \frac{\partial}{\partial x^k} g\right) + (g, Ag),$$

where A is a linear differential operator of first order. The first term on the right will contribute an integral over the boundary $M_1 \cup M_2$ of W. Near M_2, take coordinates $x^1,\dots, x^N$ as in the proof of Lemma 4.4, so that $M_2 = (x^N)^{-1}(0)$, and so that x^i is constant along the normal to M_2 for $i < N$. In these coordinates, $g^{iN} = g_{iN} = 0$ for $i < N$. It is not hard to see that $g_{\text{norm}} = 0$ on M_2 for C^∞ g implies that

$(dg)_{\text{norm}} = (\partial/\partial x^N) g_{\tan} = (\partial/\partial x^N) g$ on M_2. Hence the contribution of the first term on M_2 is

$$\int_{M_2} g^{NN} g \wedge * \frac{\partial}{\partial x^N} g = \int_{M_2} g \wedge *dg.$$

Using a dual argument near M_1, we arrive at

$$(dg, dg) + (\delta g, \delta g) = (g, g)_1 + (g, Ag).$$

But the Schwarz inequality implies

$$\begin{aligned} |(g, Ag)| &\leqslant C_1(g, g)_0 + C_2((g, g)_0 (g, g)_1)^{1/2} \\ &\leqslant \tfrac{1}{2}(g, g)_1 + (C_1 + \tfrac{1}{2}C_2{}^2)(g, g)_0, \end{aligned}$$

which yields the coerciveness inequality with $c_1 = \frac{1}{2}$, $c_2 = (C_1 + \frac{1}{2}C_2{}^2)$.

The fact that $g_{\text{norm}} = 0$ on M_2 for $C^\infty g$ implies $(dg)_{\text{norm}} = (\partial/\partial x^N) g_{\tan}$ on M_2 (and the dual statement at M_1) make it clear that V is the closure in $H_1(W, O)$ of $\mathscr{D}^q(W, O)$. Since for g in $\mathscr{D}^q(W, O)$,

$$-(g, \Delta g) \geqslant c_1(g, g)_1 - c_2(g, g)_0,$$

the convergence of g_n and Δg_n in L^2 norm imply that g_n is a Cauchy sequence in V. In particular, Gf is in V and

$$(dg, dGf) + (\delta g, \delta Gf) = (g, f), \qquad g \in V.$$

It follows [1, Theorem 9.8] that for f in $H_m(W, O)$, Gf is in $H_{m+2}(W, O)$ and in fact

$$\| Gf \|_{m+2} \leqslant K \| f \|_m \tag{5.8}$$

for some constant K. In particular, if f is C^∞, then Gf is C^∞ and satisfies $\Delta Gf = -f$, $(Gf)_{\tan} = 0$ on M_1, $(Gf)_{\text{norm}} = 0$ on M_2.

Finally, for every g in V,

$$\int_{M_2} g \wedge *dGf - \int_{M_1} \delta Gf \wedge *g = 0.$$

Since $g_{\tan}$ can be chosen arbitrarily on M_2 and g_{norm} on M_1, this implies that $(dGf)_{\text{norm}} = 0$ on M_2, $(\delta Gf)_{\tan} = 0$ on M_1.

Suppose next that f is C^∞, orthogonal to $\mathscr{H}^q$, and satisfies also

$f_{\tan} = 0$ on M_1, $f_{\text{norm}} = 0$ on M_2. On such forms, clearly, G commutes with d and δ, and we can write

$$\begin{aligned} f &= -\Delta G f \\ &= dG\delta f + \delta G d f. \end{aligned}$$

$g_1 = G\delta f = \delta G f$ is in $\mathscr{L}_{q-1}$ and $g_2 = Gdf = dGf$ is in $\mathscr{L}^{q+1}$. But g_1 also satisfies $(g_1)_{\tan} = 0$ on M_1, and this implies $(dg_1)_{\tan} = 0$ on M_1 since g_1 is C^∞; in other words, dg_1 is in $\mathscr{L}^q$. Similarly, δg_2 is in $\mathscr{L}_q$. If f satisfies the boundary conditions above but is not orthogonal to $\mathscr{H}^q$, we obtain the Hodge decomposition by applying the above to $f - h$, where $h = \lim_{t\to\infty} P(t) f$ is the orthogonal projection of f on $\mathscr{H}^q$.

Note, finally, that for the Hodge decomposition $f = dg_1 + \delta g_2 + h$ as constructed above, (5.8) implies for instance

$$\| dg_1 \|_m \leqslant K \| f \|_m \tag{5.9}$$

for a constant $K \cdot K_m$. This fact will find application in the proof of Proposition 6.4.

We turn to the proof of Proposition 5.3. To start with, embed W, as in [2], in a closed C^∞ Riemannian manifold W'. The vector bundle $E(O)$ can be extended to W' and we can define the space $\mathscr{D}(W', O)$ of C^∞ forms on W' with values in the extended bundle. We will first construct the fundamental solution of the heat equation for forms in $\mathscr{D}(W', O)$.

To do this, let $r(x, y)$ be the geodesic distance between the points x, y of W', which is defined for x, y sufficiently close, say for $r(x, y) \leqslant \delta$. Using the differential equation for geodesics, one can derive (see [4, Section 4], for instance)

$$\sum g^{ik}(x) \frac{\partial}{\partial x^i} r^2(x, y) \frac{\partial}{\partial x^k} r^2(x, y) = 4r^2(x, y),$$

$$\sum g^{ik}(x) \frac{\partial^2}{\partial x^i \, \partial x^k} r^2(x, y) = 2N + O(r(x, y)).$$

It follows from these that for ψ a function of one real variable and f a differential form, for y fixed in W',

$$\begin{aligned} \Delta(\psi(r^2(\cdot, y)) f) = {} & \psi \Delta f \\ & + \psi' \left(2N + 4B + 4r \frac{\partial}{\partial r}\right) f \\ & + 4r^2 \psi'' f, \end{aligned}$$

where $\partial/\partial r$ means the directional derivative of each component of f along the geodesic from y, and B is a matrix function, vanishing at y, which acts on the components of f.

Still keeping the point y fixed, let $\Phi_{-1}(\cdot, y) = 0$, and for $k = 0,\ldots, n+1$, let Φ_k be a double form vanishing for $r(x, y) \geqslant 2\delta$ and satisfying

$$\left(r\frac{\partial}{\partial r} + B + k\right)\Phi_k(\cdot, y) = \Delta\Phi_{k-1}(\cdot, y), \qquad r(\cdot, y) \leqslant \delta,$$

$$(\Phi_0(y, y))_{i_1\cdots i_q; j_1\cdots j_q} = g_{i_1 j_1}(y) \cdots g_{i_q j_q}(y).$$

Set

$$p(x, y, t) = (4\pi t)^{-N/2}\, e^{-r^2(x,y)/4t},$$

and

$$Q'(x, y, t) = p(x, y, t) \sum_0^{n+1} t^k \Phi_k(x, y).$$

Then Q' is a C^∞ double form on $W' \times W'$ which satisfies, for $r(x, y) \leqslant \delta$,

$$\left(\Delta - \frac{\partial}{\partial t}\right) Q' = t^{n+1} p \Delta\Phi_{n+1}\,. \tag{5.10}$$

Q' is the parametrix for the closed manifold W'. It has the properties: For any continuous form f on W',

$$Q'(t)f = \int_{W'} Q'(\cdot, y, t) \wedge {}^*f(y)$$

is a C^∞ form satisfying

$$\lim_{t\to 0} Q'(t)f = f, \qquad \text{uniformly;}$$

For $f_t = f(\cdot, t)$ C^1 on $W' \times [0, \infty)$, $\int_0^t Q'(t - t') f_{t'}\,, dt'$ is C^2 on W' for each $t > 0$, C^1 as a function of $t > 0$, and

$$\left(\Delta - \frac{\partial}{\partial t}\right) \int_0^t Q'(t - t') f_{t'}\, dt' = -f_t + \int_0^t \left(\Delta - \frac{\partial}{\partial t}\right) Q'(t - t') f_{t'}\, dt'.$$

The proofs reduce to fairly standard formulas for the heat kernel in euclidean space if one uses geodesic coordinates. The details can be gained from [3, Chapter 1].

Because of the last relation above, it follows that the kernel

$$
\begin{aligned}
P'(t) &= P'(\cdot, \cdot, t) \\
&= Q'(t) + \int_0^t Q'(t - t')\, U_{t'}\, dt'
\end{aligned}
\tag{5.11}
$$

will satisfy the heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) P'(t) = 0$$

if U_t is C^1 on $W' \times [0, \infty)$ and satisfies the integral equation

$$U_t = \left(\Delta - \frac{\partial}{\partial t}\right) Q'(t) + \int_0^t \left(\Delta - \frac{\partial}{\partial t}\right) Q'(t - t)\, U_{t'}\, dt'.$$

The first property above of the parametrix shows that P' is the fundamental solution of (5.1) on W'. Uniqueness of the fundamental solution follows by standard methods which we outline below.

Because of the way we have constructed the parametrix, the integral equation for U_t is quite easy to solve by iteration. In fact, if

$$
\begin{aligned}
U_t^{(1)} &= \left(\Delta - \frac{\partial}{\partial t}\right) Q'(t), \\
U_t^{(m+1)} &= \int_0^t \left(\Delta - \frac{\partial}{\partial t}\right) Q'(t - t')\, U_{t'}^{(m)}\, dt',
\end{aligned}
$$

then use of geodesic coordinates and standard calculations in euclidean space shows that, because of (5.10), each component of $U_t^{(m)}$ is dominated by a fixed multiple of $((mn)!)^{-1}\, t^{m(n+2)-1}k$, with k given by (5.2). Hence

$$U_t = \sum_1^\infty U_t^{(m)}$$

gives us a solution of the integral equation. It is not hard to see that the kernel U_t is a C^{2n} double form on $W' \times W'$; in particular, the kernel $P'(t)$ given by (5.11) satisfies the heat equation.

The fundamental solution is unique in the following sense. Suppose P' and P'' are kernels satisfying

$$\left(\Delta - \frac{\partial}{\partial t}\right) P'(t) = \left(\Delta - \frac{\partial}{\partial t}\right) P''(t) = 0,$$
$$\lim_{t\to 0} P'(t) f = \lim_{t\to 0} P''(t) f = f,$$

for each continuous form f. Then applying Green's theorem on the closed manifold W',

$$\begin{aligned} 0 &= \int (\Delta P'(u, y, t') \wedge {*P''}(u, x, t - t') - P'(u, y, t') \wedge {*\Delta P''}(u, x, t - t')) \\ &= \int \left(\frac{\partial}{\partial t'} P'(u, y, t') \wedge {*P''}(u, x, t - t') + P'(u, y, t') \wedge {*} \frac{\partial}{\partial t'} P''(u, x, t - t')\right) \\ &= \frac{\partial}{\partial t'} \int P'(u, y, t') \wedge {*P''}(u, x, t - t'). \end{aligned}$$

Integrating this equation over $(0, t)$ yields

$$0 = P'(x, y, t) - P''(y, x, t),$$

and taking $P'' = P'$,

$$\begin{aligned} P'(x, y, t) &= P'(y, x, t) \\ &= P''(x, y, t). \end{aligned}$$

Uniqueness and the fact that Δ commutes with d and δ show that $P'(t)$ commutes with d and δ.

Calculations of the type used in the proof of Theorem 8, Chapter 9 of [3] can be applied to show that $P'(t)$ is actually C^∞ on $W' \times W'$ and to derive estimates of the form (5.4) for the derivatives. Here is an alternative proof using the special form of the parametrix. It follows from (5.10) and (5.11) that for x fixed in W', $P'(x, y, t)$ is C^{2n} in y for each $t > 0$, and satisfies estimates of the form (5.4) for derivatives in y up to order $2n$. But uniqueness implies the semi-group property $P'(t) = P'(t/2)\, P'(t/2)$, and so

$$\begin{aligned} P'(x, y, t) &= \int P'(x, u, t/2) \wedge {*P'}(u, y, t/2) \\ &= \int P'(u, x, t/2) \wedge {*P'}(u, y, t/2), \end{aligned}$$

from which we can derive estimates for derivatives in both variables.

Finally, (5.11) implies that $P'(t)$ has a local asymptotic expansion of the form (5.5).

We now turn to the construction of the fundamental solution P of the initial boundary-value problem (5.1), using the method of single layer potentials as in Chapter 5 of [3]. A single layer potential is given by the action of a kernel on a smooth form defined on the boundary of W, producing a form on W which satisfies the heat equation with vanishing initial data in the interior, and which has an identifiable discontinuity at the boundary. Thus the addition of such a potential to P' will yield a kernel satisfying the boundary conditions 3.2 if the corresponding form on the boundary satisfies a suitable integral equation. Rather than writing down this integral equation, we find it more efficient simply to exhibit the solution which one obtains by iteration.

LEMMA 5.12. *Let P' be the fundamental solution of the heat equation on the closed manifold W' containing W. Set*

$$Q^{(0)}(x, y, t) = P'(x, y, t),$$

$$\begin{aligned} Q^{(m+1)}(x, y, t) = &- \int_0^t dt' \int_{M_1} (\delta Q^{(m)}(u, y, t - t') \wedge *P'(x, u, t') \\ &+ Q^{(m)}(u, y, t - t') \wedge *dP'(x, u, t')) \\ &+ \int_0^t dt' \int_{M_2} (P'(x, u, t') \wedge *dQ^{(m)}(u, y, t - t') \\ &+ \delta P'(x, u, t') \wedge *Q^{(m)}(u, y, t - t')), \end{aligned}$$

for $m = 0, 1, \ldots$, where all operations in the integrand on the right are applied to the variable u.

For y fixed in the interior of W, $Q^{(m)}$ is C^∞ in the interior of W and satisfies

$$\left(\Delta - \frac{\partial}{\partial t}\right) Q^{(m)}(x, y, t) = 0,$$

$$\lim_{t \to 0} Q^{(m)}(x, y, t) = 0, \qquad m \geqslant 1.$$

For $m \geqslant 1$, $Q^{(m)}$ satisfies the "jump relation"

$$\lim_{x \to x_0} Q^{(m)}(x, y, t) = Q^{(m)}(x_0, y, t) + \tfrac{1}{2}(Q^{(m-1)}(x_0, y, t))_{\tan},$$

$$\lim_{x \to x_0} \delta Q^{(m)}(x, y, t) = \delta Q^{(m)}(x_0, y, t) + \tfrac{1}{2}(\delta Q^{(m-1)}(x_0, y, t))_{\tan},$$

as x approaches the point x_0 of M_1 along the interior normal to M_1, while

$$\lim_{x\to x_0} Q^{(m)}(x, y, t) = Q^{(m)}(x_0, y, t) + \tfrac{1}{2}(Q^{(m-1)}(x_0, y, t))_{\text{norm}},$$

$$\lim_{x\to x_0} dQ^{(m)}(x, y, t) = dQ^{(m)}(x_0, y, t) + \tfrac{1}{2}(dQ^{(m-1)}(x_0, y, t))_{\text{norm}},$$

for normal approach to the point x_0 of M_2 from within W.

Finally, in a given coordinate system, the components of $Q^{(m)}$ for $m \geqslant 1$ satisfy the estimates

$$\left| \frac{\partial^n}{(\partial x^i)^n} Q^{(m)}_{i_1\cdots i_q; j_1\cdots j_q}(x, y, t) \right| \leqslant C^m(\Gamma(m/2))^{-1}\, t^{-n/2} e^{-c(D^2(x)+D^2(y))/t} k(x, y, t) \tag{5.13}$$

for $n = 0, 1$, $0 < t < t_0$, where $D(x)$ denotes the distance of the point x from the boundary $M_1 \cup M_2$ of W, and where the constants C and c, as well as the generic constants in the definition (5.2) *of k depend only on t_0 and the choice of the coordinate system.*

COROLLARY 5.14. *The kernel P defined by*

$$P(x, y, t) = \sum_{m=0}^{\infty} (-2)^m Q^{(m)}(x, y, t)$$

is the fundamental solution of the initial boundary-value problem (5.1), *and has the properties stated in Proposition* 5.3.

Proof. Because of the estimate (5.13) the series defining P converges uniformly on W and is differentiable term by term. Thus application of the jump relation to each term shows that P satisfies the relative boundary conditions of Definition 3.2 on M_1 and the absolute boundary conditions on M_2. The fact that each term $Q^{(m)}$ satisfies the heat equation with vanishing initial data in the interior of W shows then that P is indeed the fundamental solution of (5.1).

Because of the boundary conditions (3.2), we may apply Green's theorem to prove that the fundamental solution P is unique and symmetric, just as was done previously for P'. As a corollary of uniqueness, we see that P' commutes with d and δ on $\mathscr{D}$. This yields the first statement of the second paragraph of 5.3, which can also be derived directly from the definition of $Q^{(m)}$.

A further corollary of uniqueness, as before, is the semigroup property

$$P(x, y, t) = \int_W P(x, u, t/2) \wedge {}^*P(u, y, t/2)$$
$$= \int_W P(x, u, t/2) \wedge {}^*P(y, u, t/2).$$

The estimate (5.13) implies (5.4) for $n = 0$. But applying this to each factor in the integral on the right yields (5.4) for $n = 1$ as well. Although it becomes more difficult to obtain sharp estimates for higher derivatives, it is easy to prove by iteration of the semigroup property that $\Delta^n P$ is dominated by a multiple of $t^{-n}k$ for each integer n. Applying Sobolev's inequality, we see that P is C^∞ on W.

The proof of (5.13) proceeds, of course, by induction, and we will carry along an additional statement in the induction, namely, an estimate on the boundary for the components of $\underset{\sim}{Q}^{(m)}$ which actually occur in the integral defining $\underset{\sim}{Q}^{(m+1)}$ and their tangential derivatives up to order two. This will enable us to obtain the estimate (5.13) in the interior for $\underset{\sim}{Q}^{(m+1)}$ and its first tangential derivatives. It will also imply that (5.13) (with $n = 1$) holds for $d\underset{\sim}{Q}^{(m+1)}$ and $\delta\underset{\sim}{Q}^{(m+1)}$, i.e., the components of these forms have the same bounds as the tangential derivatives. Since the derivative in the normal direction to the boundary can be expressed as a linear combination of d, δ and the tangential derivatives, we will thus obtain (5.13) for all first derivatives.

We will present the details of the calculations only for the second of the four terms in the formula for $\underset{\sim}{Q}^{(m+1)}$. For this term, the suitable induction hypothesis is as follows: Take a local chart $(x^1,..., x^N)$ near the boundary M_1 as in the proof of Lemma 4.4, so that $M_1 = (x^N)^{-1}(0)$ and $g^{iN} = g_{iN} = 0$ for $i < N$. Then for x in M_1 and $i_\mu < N$, $1 \leqslant \mu \leqslant q$,

$$|(\partial_{\tan})^n \underset{\sim}{Q}^{(m)}_{(i);(j)}(x, y, t)| \leqslant C^m \left(\Gamma\left(\frac{m+1}{2}\right)\right)^{-1} t^{(m-n)/2} e^{-cD^2(y)/t} k(x, y, t) \quad (5.15)$$

for $n = 0, 1, 2$, $0 < t < t_0$, where $\partial_{\tan} = \sum_{k<N} a_k(\partial/\partial x^k)$ denotes an arbitrary derivative in a direction tangent to the boundary. To start the induction, note that $\underset{\sim}{Q}^{(0)} = P'$ satisfies (5.15) for $m = 0$, since $D(y)$ is dominated by a multiple of $r(x, y)$.

In estimating the first term of the formula for $\underset{\sim}{Q}^{(m+1)}$ we would have to carry along an estimate like (5.15) for the components of $\delta\underset{\sim}{Q}^{(m)}$, and their first tangential derivatives. We will see from the formula for $\delta\underset{\sim}{Q}^{(m+1)}$ given below that these can be obtained in exactly the same way as (5.15). The terms involving integrals over M_2 are of course dual to the first two terms.

The second term, which we will now consider, is given by

$$-\int_0^t dt' \int_{M_1} \underset{\sim}{Q}^{(m)}(u, y, t-t') \wedge {}^*dP'(x, u, t') = \int_0^t dt' F(x, y, t', t-t'),$$

where, in the coordinates we have chosen near M_1, F is given by

$$\begin{aligned} F_{(i);(j)}(x, y, t', t-t') = \frac{1}{q!} \int \cdots \int du^1 \cdots du^{N-1} \sqrt{g(u)} \\ \times \sum_{k_\mu, l_\nu < N} \prod_1^N g^{k_\mu l_\mu}(u)\, g^{NN}(u)\, Q^{(m)}_{(l);(j)}(u, y, t-t') \\ \times (dP')_{(i);N,(k)}(x, u, t'). \end{aligned}$$

The other terms can clearly be handled by similar methods.

Consider first a term F' in the form F which involves the components

$$\frac{\partial}{\partial u^k} P'_{i_1 \cdots i_q; N, k_1 \cdots k_{q-1}}, \qquad k < N,$$

of dP'. We have the estimate

$$\begin{aligned} \left| \frac{\partial}{\partial u^k} P'_{i_1 \cdots i_q; N, k_1 \cdots k_{q-1}}(x, u, t') \right| &= O\left(\frac{r(x, u)}{t'} \Phi_0(x, u) + 1 \right) p(x, u, t') \\ &\leqslant \begin{cases} (t')^{-1/2}\, k(x, u, t'), & i_\mu \text{ arbitrary}, \\ k(x, u, t'), & \text{if } \; i_\mu < N, \quad 1 \leqslant \mu \leqslant q, \end{cases} \end{aligned}$$

since the factor Φ_0 in the definition of the parametrix on W' vanishes at $u = x$ under the second condition.

The components of $\underset{\sim}{Q}^{(m)}$ which occur in the integral defining F are just those to which (5.15) applies. Hence we can use the properties of the euclidean heat kernel in R^{N-1} to obtain

$$| F'_{(i);(j)}(x, y, t', t-t') | \leqslant C^m \left(\Gamma\left(\frac{m+1}{2} \right) \right)^{-1} (t-t')^{(m-1)/2} (t')^{-1}\, k_1(x, y, t),$$

where

$$k_1(x, y, t) = e^{-c(D^2(x) + D^2(y))/t}\, t^{1/2} k(x, y, t).$$

But we will use this bound only for $t' > t/2$. When $t' < t/2$, we can integrate by parts in the integral defining F', since only the components $(\partial/\partial u^k)P'$ for $k < N$ are involved. Then using (5.15) with $n = 1$, we obtain

$$|F'_{(i);(j)}(x, y, t', t - t')| \leqslant C^m \left(\Gamma\left(\frac{m+1}{2}\right)\right)^{-1} (t - t')^{(m-2)/2}(t')^{-1/2}\, k_1(x, y, t).$$

Taken together, these two estimates form an integrable upper bound for F' on $0 < t' < t$, and we see that the contribution of F' to $\underset{\sim}{Q}^{(m+1)}$ satisfies (5.13) with $n = 0$ if the constant C has been chosen large enough (independently of m).

Since (5.15) holds also for $n = 2$, we would like to apply the same techniques to obtain (5.13) for $n = 1$. When $t' < t/2$ after the first integration by parts performed above, the integral defining F' will contain the factor

$$\frac{\partial}{\partial x^k} P'(x, u, t') = O\left(\frac{1}{t'} \frac{\partial}{\partial x^k} r^2(x, u)\right) p(x, u, t').$$

But the geodesic distance clearly satisfies

$$r^2(x, u) = \sum g_{ij}(x^i - u^i)(x^j - u^j) + O(r^3),$$

and since r^2 is C^∞ on $W' \times W'$, we can differentiate this estimate to get

$$\frac{\partial}{\partial x^k} r^2(x, u) + \frac{\partial}{\partial u^k} r^2(x, u) = O(r^2(x, u))$$

and hence

$$\left| \frac{\partial}{\partial x^k} P'(x, u, t') + \frac{\partial}{\partial u^k} P'(x, u, t') \right| \leqslant k(x, u, t').$$

Using this and then integrating by parts a second time, we gain the desired estimate.

If $i_\mu < N$, $1 \leqslant \mu \leqslant q$, we have

$$|F'_{(i);(j)}(x, y, t', t - t')| \leqslant C^m \left(\Gamma\left(\frac{m+1}{2}\right)\right)^{-1} (t - t')^{(m-1)/2}(t')^{-1/2}\, k_1(x, y, t).$$

Since

$$\int_0^t (t - t')^{(m-1)/2}(t')^{-1/2}\, dt' = t^{m/2} B\left(\frac{m+1}{2}, \frac{1}{2}\right),$$

the contribution of F' to $\underset{\sim}{Q}^{(m+1)}$ satisfies (5.15) with $n = 0$. We are dealing with smooth functions, and so we can differentiate the bound for $(\partial/\partial u^k)P'$ which yielded this estimate. Hence there is no difficulty in twice using the technique of the preceding paragraph and integrating by parts to obtain the estimate (5.15) with $n = 1, 2$ for these terms.

Now consider the form F'' which arises in F from the components

$$\frac{\partial}{\partial u^N} P'_{(i);(k)}$$

of dP'. Differentiate the estimate

$$r^2(x, u) = g_{NN}(x)(x^N - u^N)^2 + \sum_{i,j<N} g_{ij}(x)(x^i - u^i)(x^j - u^j) + O(r^3)$$

to get

$$\frac{\partial}{\partial u^N} r^2(x, u) = -2g_{NN}(x)\, x^N + O(r^2)$$

for u in M_1, as in Lemma 3.4 of [2]. Since the principal term in $(\partial/\partial u^N)P'$ is

$$\left(\frac{\partial}{\partial u^N} p(x, u, t')\right) \Phi_0(x, u) = -\frac{1}{4t'} \frac{\partial}{\partial u^N} r^2(x, u)\, p(x, u, t')\, \Phi_0(x, u),$$

we have

$$\begin{aligned} \frac{\partial}{\partial u^N} P'_{(i);(k)}(x, u, t') &= \prod_1^q g_{i_\mu k_\mu}(u)\, g_{NN}(u) \frac{x^N}{2t'} p(x, u, t') \\ &\quad + O\left(1 + \frac{r^2(x, u)}{t'}\right) p(x, u, t') \\ &\leqslant \left(1 + \frac{x^N}{t'}\right) k(x, u, t'). \end{aligned} \tag{5.16}$$

Therefore,

$$\begin{aligned} &|F''_{(i);(j)}(x, y, t', t - t')| \\ &\quad \leqslant C^m \left(\Gamma\left(\frac{m+1}{2}\right)\right)^{-1} (t - t')^{(m-1)/2} (t')^{-1/2} \left(1 + \frac{x^N}{t'}\right) e^{-c(x^N)^2/t'} k_1(x, y, t), \end{aligned}$$

and (5.13) for $n = 0$ follows from integration over $0 < t' < t$. If x is in M_1, then $x^N = 0$ and we get (5.15) for $n = 0$ just as before. And again, since differentiation of the estimate (5.16) is valid, we can obtain the bounds (5.13) and (5.15) for the tangential derivatives by the same techniques as before.

Thus we have derived (5.13) and (5.15) for the tangential derivatives of the term

$$\int_0^t dt' \int_{M_1} Q^{(m)}(u, y, t-t') \wedge {}^*dP'(x, u, t'),$$

and the same methods clearly give these estimates for the other terms in the formula for $Q^{(m+1)}$.

Before proceeding to estimate the normal derivative, we will prove the first of the four jump relations given in Lemma 5.12. This clearly arises from the term (5.16) of the kernel. Since $k_\mu < N$ for each of the indices $k_1, \ldots, k_N$ in (5.16), the principal term of that estimate will vanish at $u = x$ unless also $i_\mu < N$, $1 \leqslant \mu \leqslant N$. Hence the normal component of $Q^{(m+1)}$ will be continuous as x approaches M_1. When $i_\mu < N$, $1 \leqslant \mu \leqslant q$, if x is in W and x_0 is in M_1, with $x^i = x_0{}^i$, $i < N$, then

$$Q^{(m+1)}_{(i);(j)}(x, y, t) - Q^{(m+1)}_{(i);(j)}(x_0, y, t) = \int_0^t dt' \frac{x^N}{2t'} \int \cdots \int du^1 \cdots du^{N-1} \sqrt{g(u)}$$
$$\times Q^{(m)}_{(i);(j)}(u, y, t-t')\, p(x, u, t') + O(1)$$

as $x^n \to 0$. Using the estimate for $r^2(x, u)$ given above, one can calculate

$$\int_0^t dt' \frac{x^N}{2t'} \int \cdots \int du^1 \cdots du^{N-1} \sqrt{g(u)}\, p(x, u, t') = \tfrac{1}{2} + O(1)$$

as $x^n \to 0$, while for each $\delta > 0$,

$$\int_0^t dt' \frac{x^N}{2t'} \int \cdots \int_{r(x_0, u) + t' > \delta} du^1 \cdots du^{N-1} \sqrt{g(u)}\, p(x, u, t') = O(1).$$

Since $Q^{(m)}$ is continuous in the pair (u, t), these estimates combine to give the jump relation

$$\lim_{x^N \to 0} (Q)^{(m+1)}_{(i);(j)}(x, y, t) - Q^{(m+1)}_{(i);(j)}(x_0, y, t)) = \tfrac{1}{2} Q^{(m)}_{(i);(j)}(x_0, y, t),$$

when $i_\mu < N$, $1 \leqslant \mu \leqslant q$. Note that the third of the jump relations is just the dual statement to this.

In the interior of W, the formula for $\underset{\sim}{Q}^{(m+1)}$ can be differentiated to obtain

$$\delta \underset{\sim}{Q}^{(m+1)}(x, y, t) = -\int_0^t dt' \int_{M_1} \delta \underset{\sim}{Q}^{(m)}(u, y, t-t') \wedge {}^*dP'(x, u, t')$$
$$+ \int_0^t dt' \int_{M_2} (\delta dP'(x, u, t') \wedge {}^*\underset{\sim}{Q}^{(m)}(u, y, t-t')$$
$$+ dP'(x, u, t') \wedge {}^*d\underset{\sim}{Q}^{(m)}(u, y, t-t')),$$

$$d\underset{\sim}{Q}^{(m+1)}(x, y, t) = -\int_0^t dt' \int_{M_1} (\delta \underset{\sim}{Q}^{(m)}(u, y, t-t') \wedge {}^*\delta P'(x, u, t')$$
$$+ \underset{\sim}{Q}^{(m)}(u, y, t-t') \wedge {}^*d\delta P'(x, u, t')$$
$$+ \int_0^t dt' \int_{M_2} \delta P'(x, u, t') \wedge {}^*d\underset{\sim}{Q}^{(m)}(u, y, t-t'),$$

where again all operations in the integrands on the right apply to the variable u. We see at once from this that $\delta \underset{\sim}{Q}^{(m+1)}$ and $d\underset{\sim}{Q}^{(m+1)}$ satisfy the second and fourth jump relations, respectively.

A further immediate consequence of these formulas is the bound (5.13), with $n = 1$, for $\delta \underset{\sim}{Q}^{(m+1)}$ near the boundary M_1 and for $d\underset{\sim}{Q}^{(m+1)}$ near the boundary M_2. But to gain these bounds near the opposite boundaries we will have to transform the formula for $\underset{\sim}{Q}^{(m+1)}$ by an application of Green's theorem.

Proceeding by induction, suppose that $d\underset{\sim}{Q}^{(m)}$ and $\delta \underset{\sim}{Q}^{(m)}$ have the bounds (5.13), $n = 1$, and, moreover, have limits at the boundary of W. This is certainly the case for $m = 0$. Let $\underset{\sim}{Q}_1^{(m)} = \underset{\sim}{Q}^{(m)}$ if $m > 0$, $\underset{\sim}{Q}_1^{(0)} = 0$. Then using the fact that $\underset{\sim}{Q}^{(m)}$ satisfies the heat equation, with zero initial data when $m > 0$,

$$\underset{\sim}{Q}_1^{(m)}(x, y, t) = -\int_0^t dt' \frac{\partial}{\partial t'} \int_W \underset{\sim}{Q}^{(m)}(u, y, t-t') \wedge {}^*P'(x, u, t')$$
$$= \int_0^t dt' \int_W (\Delta \underset{\sim}{Q}^{(m)}(u, y, t-t') \wedge {}^*P'(x, u, t')$$
$$- \underset{\sim}{Q}^{(m)}(u, y, t-t') \wedge {}^*\Delta P(x, u, t'))$$
$$= \int_0^t dt' \int_{\partial W} \lim(\delta P'(x, u, t') \wedge {}^*\underset{\sim}{Q}^{(m)}(u, y, t-t')$$
$$+ P'(x, u, t') \wedge {}^*d\underset{\sim}{Q}^{(m)}(u, y, t-t')$$
$$- \delta \underset{\sim}{Q}^{(m)}(u, y, t-t') \wedge {}^*P'(x, u, t')$$
$$- \underset{\sim}{Q}^{(m)}(u, y, t-t') \wedge {}^*dP'(x, u, t')).$$

Taking the jump relations into account, the right side becomes

$$Q_1^{(m)}(x, y, t) = Q^{(m+1)}(x, y, t) + \tfrac{1}{2}Q_1^{(m)}(x, y, t)$$
$$+ \int_0^t dt' \int_{M_1} (\delta P'(x, u, t') \wedge {}^*\lim Q^{(m)}(u, y, t - t')$$
$$+ P'(x, u, t') \wedge {}^*\lim dQ^{(m)}(u, y, t - t'))$$
$$- \int_0^t dt' \int_{M_2} (\lim \delta Q^{(m)}(u, y, t - t') \wedge {}^*P'(x, u, t')$$
$$+ \lim Q^{(m)}(u, y, t - t') \wedge {}^*dP'(x, u, t')).$$

Differentiating this yields formulas for $dQ^{(m+1)}$ and $\delta Q^{(m+1)}$ which give the bounds (5.13), $n = 1$, near M_1 and M_2, respectively, and show also that the hypothesized limits exist. We might remark that (with a little more manipulation) the above shows that in fact $Q^{(m)}(x, y, t) = Q^{(m)}(y, x, t)$ and $d_x Q^{(m)}(x, y, t) = \delta_y Q^{(m)}(x, y, t)$ in the interior of W.

As observed at the start of the proof, the bounds for $dQ^{(m)}$, $\delta Q^{(m)}$, and the tangential derivatives of $Q^{(m)}$ imply the bounds (5.13) for the normal derivative as well. Hence we have completed the proof of Lemma 5.12, and, because of Corollary 5.14, of the Proposition 5.3 as well.

6. Variation of the Heat Kernel

In this section we will examine the behavior of the fundamental solution of (5.1) as the metric changes on W, the result being the following.

PROPOSITION 6.1. *Suppose given a family of metrics on W, indexed smoothly by a real parameter σ. Suppose that for each metric in the family, the normal direction to the boundary of W is the same.*

For each value of the parameter σ, let $\Delta(\sigma)$ be the Laplacian for the corresponding metric, and let P_σ be the fundamental solution of the initial value problem (5.1) *for this metric.*

Then $P_\sigma(t)$ depends differentiably on the parameter σ for each $t > 0$, and

$$\frac{d}{d\sigma} \operatorname{Tr} P_\sigma(t) = -t \operatorname{Tr}((\delta\alpha d - d\alpha\delta - \alpha\delta d + \alpha d\delta) P_\sigma(t)),$$

where $\alpha = *^{-1}\dot{*}$, $\dot{*}$ *being the derivative with respect to* σ *of the algebraic operator* $*$ *on forms.*

Remark. For closed manifolds, $(d/d\sigma)\operatorname{Tr} P_\sigma(t) = t \operatorname{Tr}(\dot{\Delta}(\sigma)\, P_\sigma(t))$. The above formula does not reduce to this in general, the reason being that P_σ commutes with d and δ only on the space $\mathscr{D}(W, O)$, which is not invariant under α.

Proof. Let σ, σ' be two values of the parameter of the family of metrics. For operations defined in terms of a metric on W, we will indicate which of σ or σ' is involved by the absence or presence of a prime, e.g., $\Delta = \Delta(\sigma)$, $\Delta' = \Delta(\sigma')$. Unless indicated otherwise by a subscript, all operations act on the variable u of the double forms below.

We will make use of the identity

$$\begin{aligned}\int_W f \wedge *g &= \int_W g \wedge *f \\ &= \int_W g \wedge *' (*')^{-1} *f \\ &= \int_W (*')^{-1} *f \wedge *'g.\end{aligned}$$

Now, to start the proof, write

$$\begin{aligned}&\int_0^t dt' \int_W (\Delta P(x, u, t') \wedge *P'(u, y, t - t') - P(x, u, t') \wedge *\Delta' P'(u, y, t - t')) \\ &\quad = \int_0^t dt' \frac{\partial}{\partial t'} \int_W P(x, u, t') \wedge *P'(u, y, t - t') \\ &\quad = \lim_{t' \to t} \int_W (*')^{-1} *P(x, u, t') \wedge *'P'(u, y, t - t') \\ &\qquad - \lim_{t' \to 0} \int_W P(x, u, t') \wedge *P'(u, y, t - t') \\ &\quad = (*')_y^{-1} *_y P(x, y, t) - P'(x, y, t).\end{aligned}$$

Next, put $\Delta = -\delta d - d\delta$, $\Delta' = -\delta' d - d\delta'$, and apply Green's formula

$$\int_W df \wedge *g = \int_W f \wedge *\delta g + \int_{\partial W} f \wedge *g,$$

using the identity given at the start to handle the term $\delta' d$. The result is

$$\begin{aligned}
&(*')_y^{-1} *_y P(x, y, t) - P'(x, y, t) \\
&\quad = \int_0^t dt' \int_W \{-dP(x, u, t') \wedge *dP'(u, y, t - t') \\
&\qquad - \delta P(x, u, t') \wedge *\delta P'(u, y, t - t') \\
&\qquad + d((*')^{-1} *P(x, u, t')) \wedge *dP'(u, y, t - t') \\
&\qquad + \delta P(x, u, t') \wedge *\delta' P'(u, y, t - t')\} \\
&\qquad + \int_0^t dt' \int_{\partial W} \{P'(u, y, t - t') \wedge *dP(x, u, t') \\
&\qquad - \delta P(x, u, t') \wedge *P'(u, y, t - t') \\
&\qquad - (*')^{-1} *P(x, u, t') \wedge *'dP'(u, y, t - t') \\
&\qquad + \delta' P'(u, y, t - t') \wedge *P(x, u, t')\}.
\end{aligned}$$

Since P and P' satisfy the relative boundary conditions on M_1 and the absolute boundary conditions on M_2 in their respective metrics, the first and last terms in the integral over $\partial W = M_1 \cup M_2$ vanish. The second term vanishes on M_1 and the third on M_2. But the normal direction to the boundary is the same in each metric. Hence (see the remark at the start of the proof of Lemma 5.12) there are coordinates $(x^1,..., x^N)$ near a point of the boundary M_k such that $M_k = (x^N)^{-1}(0)$, while on M_k, $g_{iN} = g'_{iN} = 0$, $i < N$. Therefore on M_2, $P'_{\mathrm{norm}} = 0$ implies $(*P')_{\tan} = 0$, while on M_1, $P_{\tan} = 0$ implies $((*')^{-1} *P)_{\tan} = 0$. So these terms also disappear, and after a little rearranging

$$\begin{aligned}
&(*')_y^{-1} *_y P(x, y, t) - P'(x, y, t) \\
&\quad = \int_0^t dt' \int_W \{-dP(x, u, t') \wedge (* - *')\, dP'(u, y, t - t') \\
&\qquad + d((*')^{-1}(* - *')\, P(x, u, t')) \wedge *'dP'(u, y, t - t') \\
&\qquad - \delta P(x, u, t') \wedge * (\delta - \delta')\, P'_\sigma(u, y, t - t')\}.
\end{aligned} \tag{6.2}$$

In a local coordinate system, the operator $* - *'$ is represented by a matrix; since the metric depends smoothly on the parameter σ, we may write

$$* - *' = (\sigma - \sigma')\, A(\sigma, \sigma'),$$

where A is represented by a bounded matrix satisfying

$$\lim_{\sigma' \to \sigma} A(\sigma, \sigma') = \dot{*}(\sigma)$$

uniformly on W. We have the estimate (5.4) for each of dP and dP' and applying the standard calculations for the heat kernel in euclidean space, we see that each component of the first term of the volume integral above is bounded by a fixed multiple of $(\sigma - \sigma')(t'(t - t'))^{-1/2}$. Hence the contribution of the first term to the right side is $O(\sigma - \sigma')$.

The derivatives of the matrix A in a fixed local coordinate system on W are likewise bounded and continuous in σ'. Hence the estimate (5.4) can be applied also to the terms which involve

$$\begin{aligned} d((*')^{-1}(* - *')\, P) &= (\sigma - \sigma')\, d((*')^{-1} AP), \\ (\delta - \delta')\, P &= (\sigma - \sigma')(A *^{-1} \delta + \delta(*')^{-1} A)\, P. \end{aligned}$$

We see that their contribution to the right side is also $O(\sigma - \sigma')$.

Applying the bounded operator $*_y'$ to both sides of (6.2), we see that $*P - *'P'$ is $O(\sigma - \sigma')$ uniformly on W. In particular, $P' = P_{\sigma'}$ is continuous in σ', uniformly on W, and we can use this fact and the uniform continuity of A and its derivatives to obtain

$$\begin{aligned} \frac{d}{d\sigma}(*_y(\sigma)\, P_\sigma(x, y, t) &= \lim_{\sigma' \to \sigma} (\sigma - \sigma')^{-1}(*_y P(x, y, t) - *_y' P'(x, y, t)) \\ &= \int_0^t dt' \int_W *_y\{-dP(x, u, t') \wedge \dot{*} dP(u, y, t - t') \\ &\quad + d(*^{-1} \dot{*}\, P(x, u, t')) \wedge *dP(u, y, t - t') \\ &\quad - \delta P(x, u, t') \wedge ** \dot{*}\, *^{-1} \delta P(u, y, t - t') \\ &\quad - \delta P(x, u, t') \wedge *\delta(*^{-1} \dot{*}\, P(u, y, t - t'))\}. \end{aligned}$$

We see that $(d/d\sigma) * P$ satisfies the same estimate (5.4) as does P. But now we can again apply Green's formula, taking care that the boundary conditions are satisfied, to rewrite the volume integral above:

$$\begin{aligned} \frac{d}{d\sigma}(*_y(\sigma)\, P_\sigma(x, y, t)) = \int_0^t dt' \int_W & *_y\{-P(x, u, t') \wedge *\delta *^{-1} \dot{*}\, dP(u, y, t - t') \\ & + *^{-1} \dot{*}\, P(x, u, t') \wedge *\delta dP(u, y, t - t') \\ & - P(x, u, t') \wedge *d\, \dot{*}\, *^{-1} \delta P(u, y, t - t') \\ & - *^{-1} \dot{*}\, P(u, y, t - t') \wedge d\delta P(x, u, t')\}. \end{aligned}$$

We can further transform this expression by using

$$\begin{aligned}\int_W *^{-1}\dot{*}f \wedge *g &= \int_W g \wedge \dot{*}f\\ &= \frac{d}{d\sigma}\int_W g \wedge *f\\ &= \frac{d}{d\sigma}\int_W f \wedge *g\\ &= \int_W f \wedge \dot{*}g,\end{aligned}$$

and

$$\begin{aligned}0 &= \frac{d}{d\sigma}(*^{-1}*)\\ &= (-1)^{q(N-q)}\frac{d}{d\sigma}(**)\\ &= \dot{*}*^{-1} + *^{-1}\dot{*}.\end{aligned}$$

The result is, finally,

$$\begin{aligned}\frac{d}{d\sigma}(*_y(\sigma)\,P_\sigma(x, y, t)) = &-\int_0^t dt'\int_W *_y\{P(x, u, t') \wedge *\delta\alpha dP(u, y, t-t')\\ &- P(x, u, t') \wedge *\alpha\delta dP(u, y, t-t')\\ &- P(x, u, t') \wedge *d\alpha\delta P(u, y, t-t')\\ &+ P(u, y, t-t') \wedge *\alpha d\delta P(x, u, t')\}\\ = &-\int_0^t dt'\int_W \{\delta\alpha d - \alpha\delta d - d\alpha\delta + \alpha d\delta)_u\\ &\times P(x, v, t') \wedge *_u *_y P(u, y, t-t')\}|_{v=u}\\ &+\int_0^t dt'\int_W \{(\dot{*}\,d\delta)_v - (\dot{*}\,d\delta)_u\\ &\times P(x, v, t') \wedge *_u *_y P(u, y, t-t')\}|_{v=u}\,. \qquad (6.3)\end{aligned}$$

The operation of integrating the exterior product of two forms over W does not, of course, depend on the metric. Since the trace of $P_\sigma(t)$ is given by the integral of the exterior product with itself of the double form $*_y P_\sigma(x, y, t)$, taken at $y = x$, we obtain $(d/d\sigma) \operatorname{Tr} P_\sigma(t)$ by applying the latter operation to (6.3). Having done so, we can interchange the order of integration on the right, and integrate with respect to x before applying d and δ on the variables u and v. But then each of the integrals with respect to x reduces to

$$\int_W P(x, v, t') \wedge *_x P(u, x, t - t') = P(u, v, t).$$

Hence $(d/d\sigma) \operatorname{Tr} P_\sigma(t)$ is given by the integral over W of the exterior product with itself of

$$-t(\delta\alpha d - \alpha\delta d - d\alpha\delta + \alpha d\delta)_u P(u, v, t) + t((\dot{*}\, d\delta)_v - (\dot{*}\, d\delta)_u P(u, v, t),$$

taken at $v = u$. Since this operation is the trace, and since the second term above vanishes at $v = u$, we get the desired formula for $(d/d\sigma) \operatorname{Tr} P_\sigma(t)$.

We will apply Proposition 6.1 in the next section to examine the behavior of the analytic torsion as σ vanishes. We will also want to consider the behavior of the R-torsion defined in Section 3. For this we will have to compare orthonormal bases for the spaces of harmonic forms for the two different metrics. The result we will want is the following.

PROPOSITION 6.4. *Suppose, as before, that we have a smoothly parameterized family of metrics on W, for each of which the normal direction to the boundary of W is the same.*

For each value σ of the parameter, let $\mathscr{H}_\sigma$ be the space of forms which are harmonic in the corresponding metric, and which satisfy the relative boundary conditions of 3.2 on M_1 and the absolute boundary conditions on M_2. For each σ there is an orthonormal base $(h_j(\sigma))$ of $\mathscr{H}_\sigma$ such that for each j $h_j(\sigma)$ is a differentiable function of σ, $\dot{h}_j = (d/d\sigma)h_j$ is in $\mathscr{L}^q$, and

$$(h_j, \dot{h}_j) = \tfrac{1}{2}(h_j, \alpha h_j), \tag{6.5}$$

where as before $\alpha = *^{-1} \dot{*}$.

Proof. Let f be a smooth form on W which satisfies $f_{\tan} = 0$ on M_1, $f_{\text{norm}} = 0$ on M_2. These boundary conditions are independent of the choice of the metric within our family, since the normal direction to the boundary is the same for each metric. Moreover, we saw in the proof of Proposition 6.1 that for each σ, αf satisfies the same boundary conditions.

We can therefore apply the Hodge decomposition of Corollary 5.7 to αf:

$$\alpha f = dg_1 + \delta g_2 + h.$$

Define

$$F_\sigma(f) = -dg_1 - \tfrac{1}{2}h.$$

The form $F_\sigma(f)$ so defined satisfies

$$\begin{aligned} dF_\sigma(f) &= 0, \\ \delta_\sigma F_\sigma(f) &= -\delta_\sigma(\alpha_\sigma f), \\ (F_\sigma(f), h) &= -\tfrac{1}{2}(\alpha_\sigma f, h), \qquad h \in \mathscr{H}_\sigma . \end{aligned}$$

By (5.9), F_σ is a bounded operator on the space $H_m(W, O)$, which is clearly independent of σ. This means that we can apply the Picard iteration method to solve the differential equation

$$\frac{d}{d\sigma} h(\sigma) = F_\sigma(h(\sigma))$$

in $H_m(W, O)$, with initial value $h(0)$ in $\mathscr{H}_0$ at $\sigma = 0$. The solution $h(\sigma)$ is, of course, independent of m, and since m is arbitrary $h(\sigma)$ is C^∞ on W for each σ.

Writing

$$h(\sigma) = h(0) + \int_0^\sigma F_\sigma(h(\sigma'))\, d\sigma'$$

shows that $dh(\sigma) = 0$. We will show that for g satisfying $g_{\mathrm{tan}} = 0$ on M_1 and $g_{\mathrm{norm}} = 0$ on M_2, we also have $(h(\sigma), dg) = 0$; by Corollary 5.7, this will imply that $h(\sigma)$ is in $\mathscr{H}_\sigma$.

To see this, write

$$\begin{aligned} \frac{d}{d\sigma}(h(\sigma), dg) &= \frac{d}{d\sigma}\int_W h(\sigma) \wedge {}^*dg \\ &= \left(\frac{d}{d\sigma} h(\sigma), dg\right) + (h(\sigma), \alpha dg). \end{aligned}$$

But

$$\begin{aligned}\left(\frac{d}{d\sigma} h(\sigma), dg\right) &= (F_\sigma(h(\sigma)), dg) \\ &= (\delta_\sigma F_\sigma(h(\sigma)), g) \\ &= -(\delta_\sigma \alpha_\sigma h(\sigma), g) \\ &= -(h(\sigma), \alpha_\sigma\, dg).\end{aligned}$$

Hence $(h(\sigma), dg)$ is constant, and since $h(0)$ is in $\mathscr{H}_0$, $(h(\sigma), dg) = 0$ for all σ.

Now let $(h_j(0))$ be an orthonormal base of $\mathscr{H}_0$, and let $h_j(\sigma)$ be the solution of our differential equation with initial value $h_j(0)$. Since

$$\begin{aligned}\frac{d}{d\sigma}(h_j(\sigma), h_k(\sigma)) &= (\dot{h}_j(\sigma), h_k(\sigma)) + (h_j(\sigma), \dot{h}_k(\sigma) + (h_j(\sigma), \alpha h_k(\sigma)) \\ &= (F_\sigma(h_j(\sigma)), h_k(\sigma)) + (h_j(\sigma), F_\sigma(h_k(\sigma))) + (h_j(\sigma), \alpha H_k(\sigma))) \\ &= -\tfrac{1}{2}(\alpha h_j(\sigma), h_k(\sigma)) - \tfrac{1}{2}(h_j(\sigma), \alpha h_k(\sigma)) + (h_j(\sigma), \alpha h_k(\sigma)) \\ &= 0,\end{aligned}$$

$(h_j(\sigma))$ is orthonormal in $\mathscr{H}_\sigma$, and, of course, is a base since

$$\dim \mathscr{H}_\sigma = \dim \mathscr{H}_0 .$$

Equation (6.5) comes from setting $k = j$ in the second line above.

7. Variation of the Torsion

Let W be a manifold with boundary as described in Section 3, and O a representation of the fundamental group $\pi_1(W)$ by orthogonal matrices. Let $P_q(t)$ be the fundamental solution of the initial value problem (5.1) for forms in $\mathscr{D}^q(W, O)$.

We want to define the analytic torsion for W as in 1.6, but we do not want to assume that the homology of W with coefficients in $E(O)$ is trivial. So we must alter the Definition (1.5) of the zeta function. It is clear from the spectral representation of $P_q(t)$ that

$$H_q = \lim_{t\to\infty} P_q(t)$$

is the orthogonal projection operator onto the space $\mathscr{H}^q$ of harmonic q-forms in $\mathscr{D}^q$, and that

$$\operatorname{Tr}(P_q(t) - H_q)$$

decreases exponentially as $t \to \infty$. Hence

$$\zeta_{q,O}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(P_q(t) - H_q)\, dt \tag{7.1}$$

defines an analytic function of s for Re s sufficiently large. It is known [17] as in the case of a closed manifold, that $\zeta_{q,O}$ has a meromorphic extension which is analytic at $s = 0$.

Definition 7.2. The analytic torsion $T_W(O)$ of the Riemannian manifold W with boundary is the positive real root of

$$\log T_W(O) = \tfrac{1}{2} \sum_{q=0}^{N} (-1)^q\, q\zeta'_{q,O}(0).$$

We will investigate the change in $T_W(O)$ as the metric varies on W, leaving the normal direction to the boundary the same. The method is the same as that of the proof of Theorem 2.1, but we no longer have the asymptotic expansion (5.6) which showed that $\zeta_{q,O}(0) = 0$ for closed manifolds. We are therefore forced to consider instead of $T_W(O)$, the quotient $T_W(O)/T_W(O')$ for two representations O, O' of $\pi_1(W)$.

Theorem 7.3. *Let O, O' be representations of the fundamental group $\pi_1(W)$ such that the homology of W with coefficients in $E(O)$ and $E(O')$ is the same. Suppose a family of metrics on W, parameterized by σ, for which the normal direction to the boundary is the same. Then*

$$\frac{d}{d\sigma} \log(T_W(O)/T_W(O')) = \tfrac{1}{2} \sum_{q=0}^{N} (-1)^q \operatorname{Tr}(\alpha(H_q - H_q')),$$

where H_q and H_q' are the projections onto the spaces $\mathscr{H}_q$, $\mathscr{H}_q'$ of harmonic forms in $\mathscr{D}^q(W, O)$ and $\mathscr{D}^q(W, O')$, respectively. Recall that α is the algebraic operator $*^{-1} \dot{*}$.

Proof. Note that the assumption about the homology of W implies that, for each choice of σ,

$$\operatorname{Tr} H_q = \operatorname{Tr} H_q'$$

and that

$$|\operatorname{Tr} P_q(t) - \operatorname{Tr} P_q'(t)| \leqslant e^{-ct}, \qquad t \geqslant 1,$$

for some $c > 0$, where, of course, P_q and P_q' are the heat kernels for the spaces $\mathscr{D}^q(W, O)$ and $\mathscr{D}^q(W, O')$, respectively.

Let x be a point of W; we can assume that the neighborhood $U_\delta = \{y : r^2(x, y) < \delta\}$ of x is simply connected, where r is the geodesic distance on W. Being a local operator, the Laplacian is the same on sections of $\mathscr{D}^q(W, O)$ and $\mathscr{D}^q(W, O')$ over U_δ. This means that we can apply Green's formula as in the proof of Proposition 6.1 to obtain

$$\begin{aligned} P_q(x, y, t) - P_q'(x, y, t) = \int_0^t dt' \int_{r^2(x,u)=\delta} &\{P_q'(u, y, t-t') \wedge {*dP_q(x, u, t')} \\ &- \delta P_q(x, u, t') \wedge {*P_q'(u, y, t-t')} \\ &- P_q(x, u, t') \wedge {*dP_q'(u, y, t-t')} \\ &+ \delta P_q(u, y, t-t') \wedge {*P(x, u, t')}\}. \end{aligned} \tag{7.4}$$

Because of the boundary conditions, this holds whether or not the boundary of W intersects U_δ. Using the estimates (5.4), we have then

$$|\operatorname{Tr} P_q(t) - \operatorname{Tr} P_q'(t)| \leqslant Kt^{-N/2}e^{-c/t}, \qquad t \leqslant 1.$$

Hence

$$\zeta_{q,O}(s) - \zeta'_{q,O}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(P_q(t) - P_q'(t))\, dt$$

has the analytic extension in the s-plane given by the right side above, and

$$\log(T_W(O)/T_W(O')) = \tfrac{1}{2} \sum_{q=0}^{N} (-1)^q q \int_0^\infty \operatorname{Tr}(P_q(t) - P_q'(t)) \frac{dt}{t}. \tag{7.5}$$

Because of (7.4), we can differentiate under the integral sign on (7.5) with respect to the parameter σ. Proposition 6.1 gives

$$\begin{aligned} &\frac{d}{d\sigma} \log(T_W(O)/T_W(O')) \\ &\quad = \tfrac{1}{2} \sum_{q=0}^{N} (-1)^q q \int_0^\infty \operatorname{Tr}((\alpha\delta d - \alpha d\delta + d\alpha\delta - \delta\alpha d)(P_q(t) - P_q'(t)))\, dt. \end{aligned}$$

One must now take care in permuting the operators above, since α

does not leave the spaces $\mathscr{D}^q$ invariant. However, we can see by an application of Green's theorem that

$$\begin{aligned}\operatorname{Tr}(d\alpha\delta P_q(t)) &= \operatorname{Tr}(\alpha\delta P_q(t)\, d)\\ &= \operatorname{Tr}(\alpha\delta d P_{q-1}(t)),\\ \operatorname{Tr}(\delta\alpha d P_q(t)) &= \operatorname{Tr}(\alpha d\delta P_{q+1}(t)).\end{aligned}$$

Hence, as in the proof of Theorem 2.1,

$$\begin{aligned}2\frac{d}{d\sigma}\log(T_W(O)/T_W(O')) &= \sum_{q=0}^{N}(-1)^q\int_0^\infty \frac{d}{dt}\operatorname{Tr}(\alpha(P_q(t)-P_q'(t)))\,dt\\ &= \sum_{q=0}^{N}(-1)^q\lim_{t\to\infty}\operatorname{Tr}(\alpha(P_q(t)-P_q'(t)))\\ &= \sum_{q=0}^{N}(-1)^q\operatorname{Tr}(\alpha(H_q-H_q')).\end{aligned}$$

In 3.6 we defined the R-torsion for a triangulation K of W, and in Theorem 4, we showed that this R-torsion depends only on the manifold W and its Riemannian structure. We will now examine the behavior of the R-torsion as the metric varies.

THEOREM 7.6. *Suppose as before a family of metrics on W with the real parameter σ, for each of which the normal direction to the boundary is the same. Then the R-torsion τ_W of Definition* 3.6 *satisfies*

$$\frac{d}{d\sigma}\log\tau_W(O) = \tfrac{1}{2}\sum_{q=0}^{N}(-1)^q\operatorname{Tr}(\alpha H_q),$$

for each representation O of $\pi_1(W)$, where H_q is the projection operator onto the space of harmonic forms in $\mathscr{D}^q(W, O)$, and $\alpha = *^{-1}\dot{*}$.

Proof. Let K be a smooth triangulation of W, with subcomplexes L_1, L_2 triangulating the boundary manifolds M_1 and M_2, so that $\tau_W = \tau_{K,L_1}$. We assume K fixed; then τ_W depends on the metric only through the choice of a preferred base of the homology groups as the image of an orthonormal base of harmonic forms.

In computing $\tau_{K,L_1}(O)$, we can choose a base for the boundaries $B_q(K, L_1; O) = \partial C_{q+1}(K, L_1; O)$ arbitrarily, so we proceed as follows.

For each q, pick a base $\mathbf{b}^q = (b_j{}^q)$ of the space of coboundaries $B^q(K, L_1 ; O) = \partial^* C^{q-1}(K, L_1 ; O)$, independently of the metric. For each b_j^{q+1}, pick an element $\tilde{b}_j^{q+1}$ of $C^q(K, L_1 ; O)$ such that $\partial^* \tilde{b}_j^{q+1} = b_j^{q+1}$, again making the choice independent of σ. Finally, for each σ, let $\mathbf{h}^q = (h_j{}^q)$ be an orthonormal base of harmonic q-forms in the space $\mathscr{D}^q(W, O)$ in the corresponding metric. Then $(b_j{}^q, \tilde{b}_j^{q+1}, A^q(h_j{}^q))$ is a base for $C^q(K, L_1 ; O)$.

Let $\boldsymbol{\xi} = (\xi_\mu)$ be an orthonormal base of R^n; $\boldsymbol{\xi}$, together with the cells e of the triangulation, determines a preferred base of $C^q(K, L_1 ; O)$. Let D^q be the matrix of the change from this to the base constructed above, so that

$$D^q = \left((\xi_\mu , b_j{}^q(e)), (\xi_\mu , \tilde{b}_j^{q+1}(e)), \int_e (\xi_\mu , h_j{}^q)\right),$$

where e runs through the q-simplexes of $K - L_1$.

Consider the base of $C_q(K, L_1 ; O)$ which is dual to this. Part of this base consists of elements $b_q{}^i$ of C_q satisfying

$$\langle b_q{}^i, b_j{}^q \rangle = 0,$$

$$\langle b_q{}^i, \tilde{b}_j^{q+1} \rangle = \delta_{ij} ,$$

$$\langle b_q{}^i, A^q(h_j{}^q) \rangle = 0.$$

These elements form a base for the boundaries $B_q = \partial C_{q+1}$.

Another part of the dual base consists of elements $\tilde{b}_{q-1}^i$ of C_q satisfying

$$\langle \tilde{b}_{q-1}^i , b_j{}^q \rangle = \delta_{ij} ,$$

$$\langle \tilde{b}_{q-1}^i , \tilde{b}_j^{q+1} \rangle = 0,$$

$$\langle \tilde{b}_{q-1}^i , A^q(h_j{}^q) \rangle = 0.$$

For each such element, $\partial \tilde{b}_{q-1}^i = b_{q-1}^i$.

Finally, since

$$\langle A_q(h_i{}^q), A^q(h_j{}^q) \rangle = \delta_{ij} ,$$

$$\langle A_q(h_i{}^q), b_j{}^q \rangle = 0,$$

the remaining members of the dual base are just the elements

$$A_q(h_i{}^q) + \partial c_q{}^i,$$

where

$$\langle c_q{}^i, b_j^{q-1} \rangle = -\langle A_q(h_i{}^q), \tilde{b}_j^{q-1} \rangle.$$

and $h_i' = \underset{\sim}{Q}_z'\phi_i$, then one has $\phi_1 \otimes \bar{\phi}_2 \leftrightarrow h_1' \otimes \bar{h}_2'$ in the sense of (26) and (27), i.e.,

$$(h_1' \otimes \bar{h}_2')(u_z) = \eta_z'(\underline{u}, 0)(\phi_1 \otimes \bar{\phi}_2)(u). \tag{42}$$

To prove (42), we first assume that $h_1' = \underset{\sim}{Q}_z'\phi_1 \in \mathfrak{F}_z'(\lambda)$ for some λ, $\mathfrak{F}_z'(\lambda)$ denoting the subspace of $\mathfrak{F}_z'$ corresponding to $\mathfrak{F}_{z^*}(\lambda)$ under the isomorphism $\mathfrak{F}_z' \simeq \mathfrak{F}_{z^*}$. Then, in view of (21), (21′) and (34), the left side of (42) can be transformed as follows:

$$\begin{aligned}
(h_1' \otimes \bar{h}_2')(u_z) &= \bar{\gamma}_{z^*} \int_{W_+} h_1'(w)\, \overline{\kappa_{z^*}(u_{z^*} - w^*, 0)^{-1}}\, \kappa_z'(u_z - w, w) \\
&\quad \times \left(\int_{V_1} \overline{q_{z^*}(u_{z^*} - w^* - v_1)}\, \overline{\phi_2(v_1)}\, dv_1\right) d_z w \\
&= \bar{\gamma}_{z^*}\eta_z'(\underline{u}, 0) \int_{V_1} \left(\int_{W_+} \overline{\kappa_{z^*}(w^*, 0)^{-1}}\, \overline{q_{z^*}(w^* - v_1)}\, h_1'(w)\right. \\
&\quad \times \left.\kappa_z'(w, w)^{-1}\, d_z w\right) \overline{\phi_2(u_1 - v_1)}\, \epsilon(\langle v_1 - \tfrac{1}{2}u_1, u_2\rangle)^{-1}\, dv_1 \\
&= \eta_z'(\underline{u}, 0) \int_{V_1} \phi_1(v_1)\, \overline{\phi_2(u_1 - v_1)}\, \epsilon(\langle v_1 - \tfrac{1}{2}u_1, u_2\rangle)^{-1}\, dv_1 ,
\end{aligned}$$

which proves (42) for $h_1' \in \mathfrak{F}_z'(\lambda)$. This computation is legitimated by the absolute convergence of the double integral on the second line, which follows from the fact that the function

$$\Phi_1(v_1) = \int_{W_+} |\,\kappa_{z^*}(w^*, 0)|^{-1}\, |\,q_{z^*}(w^* - v_1)|\, |\,h_1'(w)|\, \kappa_z'(w, w)^{-1}\, d_z w$$

belongs to $L_2(V_1)$ (see 10). It follows from (42) and from what we have proved in 9 that for $h_1' \in \mathfrak{F}_z'(\lambda)$ and $h_2' \in \mathfrak{F}_z'$ one has $h_1' \otimes \bar{h}_2' \in \mathfrak{L}_z'$ and

$$\|\, h_1' \otimes \bar{h}_2' \,\|_z' = \|\, \phi_1 \,\| \cdot \|\, \phi_2 \,\| = \|\, h_1' \,\|_z' \cdot \|\, h_2' \,\|_z' . \tag{43}$$

Next, to prove (42) in the general case, let h_1' be any element of $\mathfrak{F}_z'$ and let $\{h_{1\nu}'\}$ be a sequence in $\mathfrak{F}_z'(\lambda)$ such that $\lim_{\nu\to\infty} h_{1\nu}' = h_1'$. Then by (43), $\{h_{1\nu}' \otimes \bar{h}_2'\}$ is a Cauchy sequence in $\mathfrak{L}_z'$, so that it has a limit. But, by (41), one has $\lim_\nu (h_{1\nu}' \otimes \bar{h}_2')(w) = (h_1' \otimes \bar{h}_2')(w)$ for all $w \in W_+$.

Therefore, one has $h_1' \otimes \bar{h}_2' = \lim_\nu(h'_{1\nu} \otimes \bar{h}_2') \in \mathfrak{L}_z'$ and

$$(h_1' \otimes \bar{h}_2')(u_z) = \lim_{\nu\to\infty}(h'_{1\nu} \otimes \bar{h}_2')(u_z) = \eta_z'(\underline{u}, 0) \lim_{\nu\to\infty}((P_z'h'_{1\nu}) \otimes \phi_2)(u)$$

$$= \eta_z'(\underline{u}, 0)((\lim_{\nu\to\infty} P_z'h'_{1\nu}) \otimes \phi_2)(u) = \eta_z'(\underline{u}, 0)(\phi_1 \otimes \phi_2)(u)$$

(for all $u \in V$), which completes the proof of (42). The last assertion of the Theorem is obvious.

Now, let (ψ_ν) be an orthonormal basis of $\mathfrak{F}_z'$. Then every element f' in $\mathfrak{L}_z'$ can be expressed uniquely in the form

$$f' = \sum_{\nu=1}^{\infty} \psi_\nu \otimes \bar{h}_\nu'$$

with $h_\nu' \in \mathfrak{F}_z'$. Then one obtains the following

COROLLARY. *One has*

$$\overline{h_\nu'(w)} = \int_{W_+} \overline{\psi_\nu(w')} f'(w + w')\, \kappa_z'(w + w', w')^{-1}\, d_z w'. \tag{44}$$

Proof. Put $e_w'(w') = \kappa_z'(w', w)$. Then $e_w' \in \mathfrak{F}_z'$ and for any $h' \in \mathfrak{F}_z'$ one has $h'(w) = (e_w', h')$. Therefore one has

$$\overline{h_\nu'(w)} = \overline{(e_w', h_\nu')} = (\psi_\nu \otimes \bar{e}_w', f'),$$

where

$$(\psi_\nu \otimes \bar{e}_w')(w') = \int_{W_+} \psi_\nu(w'')\, \kappa_z'(w, w' - w'')\, \kappa_z'(w' - w'', w'')\, d_z w''$$

$$= \kappa_z'(w, w') \int_{W_+} \psi_\nu(w'')\, \kappa_z'(w' - w - w'', w'')\, d_z w''$$

$$= \kappa_z'(w, w')\, \psi_\nu(w' - w).$$

Thus one has

$$\overline{h_\nu'(w)} = \int_{W_+} \overline{\kappa_z'(w, w')\, \psi_\nu(w' - w)} f'(w')\, \kappa_z'(w', w')^{-1}\, d_z w'$$

$$= \int_{W_+} \overline{\psi_\nu(w')} f'(w' + w)\, \kappa_z'(w' + w, w')^{-1}\, d_z w'. \qquad \text{Q.E.D.}$$

A section of $E(O)$ can be identified with a mapping f from $\tilde{W}$ into R^n which satisfies $f \circ g = O(g) f$, for g in $\pi_1(W)$. But for such a map $f \circ \Psi$ satisfies

$$\begin{aligned}(f \circ \Psi) \circ g &= (f \circ \rho g) \circ \Psi \\ &= (O(\rho g) f) \circ \Psi \\ &= O_u(g)(f \circ \Psi).\end{aligned}$$

Hence the restriction of f to W_u is a section of the bundle $E(O_u)$; and the space $\mathscr{D}(W_u, O_u)$ can be identified with the space of restrictions to W_u of forms in $\mathscr{D}(W, O)$ which satisfy the boundary conditions 3.2 on $M_u = \varphi^{-1}(u)$.

There is, of course, a similar identification of the chain complex $C(K_u ; O_u)$ with $C(K_u ; O)$, if K is a triangulation of W which contains a triangulation K_u of W_u as a subcomplex.

To prove Corollary 7.8, let X be the vector field dual to $d\varphi$ in the metric of W. If the interval $[u_1, u_2]$ contains no critical points, then X determines a differomorphism F_u of W_u onto W_{u_1} for $u_1 \leqslant u \leqslant u_2$, as in Section 3 of [9]. Hence we can identify W_u with W_{u_1} equipped with a new metric. Since X is normal to $M_u = \varphi^{-1}(u)$, the normal direction to $M_2 = \varphi^{-1}(u_1)$ will be the same for each of the two metrics.

Hence we need only verify that the homology of W_u with coefficients in $E(O)$ and $E(O')$ is the same. Then, because of the identification indicated in the remark above, we can apply Corollary 7.7.

It is clearly sufficient to make this verification only for $u = m + \frac{1}{2}$, $m = 0, 1, \ldots, N$. We can assume a triangulation K of W, which for each m contains a triangulation K_m of $W_{m+1/2}$ as a subcomplex. The inclusion $K_{m-1} \subset K_m$ determines the exact sequence of homology groups

$$\to H_q(K_{m-1} ; O) \to H_q(K_m ; O) \to H_q(K_m, K_{m-1} ; O) \to H_{q-1}(K_{m-1} ; O).$$

But (see Lemma 9.2 of [10]) $H_q(K_m, K_{m-1} ; O)$ is zero for $q \neq m$, and is isomorphic for $q = m$ to the tensor product of R^n and a free abelian group with one generator for each critical point of index m. In particular, $H_m(K_m, K_{m-1} ; O)$ can be identified with $H_m(K_m, K_{m-1} ; O')$.

Thus we have, from the homology exact sequence,

$$\begin{aligned}H_q(K_m ; O) &\approx H_q(K_{m+1} ; O) \\ &\approx H_q(K; O) = 0, \qquad q < m,\end{aligned}$$

by the hypothesis of Corollary 7.8. Also

$$H_q(K_m\,;O) \approx H_q(K_0\,;O) = 0, \qquad q > m,$$

and

$$H_m(K_m\,, K_{m-1}\,;O) \approx H_m(K_m\,;O) \oplus H_{m-1}(K_{m-1}\,;O).$$

But the last isomorphism holds also when O is replaced by O', and we have seen that the two left sides can be identified for each m. Hence by induction on m,

$$H_m(K_m\,;O) \approx H_m(K_m\,;O').$$

Since we have seen that we can also identify $H_q(K_m\,;O)$ with $H_q(K_m\,;O_{m+1/2})$, the hypothesis of Corollary 7.7 is satisfied and Corollary 7.8 follows.

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Printed in Belgium