Compensation - Day 4: The Equitability Procedure & Adjusted Winner Method

MA 111: Intro to Contemporary Math

November 4, 2013
Compensation Method: Knaster’s Procedure

Definition

1(a) All $N$ people secretly bid on a resource.

1(b) The highest bidder receives the resource but pays $w - \frac{w}{N}$ to the mediator to distribute.

1(c) Everyone else receives $\frac{b_i}{N}$. In words, everyone gets their fair share!

The amount of money leftover is called the **surplus** and might be more than zero.

2 Everyone (including the winner) receives an equal portion of the surplus.

If the surplus is positive, then everyone gets extra money in addition to what was received above in Step 1.

Knaster’s method is also sometimes called **The Method of Sealed Bids**.
Our Average Young Ladies from before decide to use Knaster’s Procedure for The Burn Book.

<table>
<thead>
<tr>
<th></th>
<th>Regina</th>
<th>Gretchen</th>
<th>Karen</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The Burn Book</strong></td>
<td>$300</td>
<td>$210</td>
<td>$240</td>
</tr>
</tbody>
</table>

From Step 1 of Knaster’s, Regina gets the book but pays $300 - $\frac{300}{3} = $200. Gretchen and Karen each receive their fair share.

- Finish Knaster’s Procedure and find $x_{Regina}$, $x_{Gretchen}$, $x_{Karen}$.
- Do each of the Average Young Ladies get their fair share?
- Compare $\frac{b_{Regina}}{x_{Regina}}$ with $\frac{b_{Gretchen}}{x_{Gretchen}}$. What do you notice?
Compensation Method: Equitability Procedure

1(a) Every person (all $N$ of them) secretly bid on a resource.

1(b) The highest bidder receives the resource and pays $w - \frac{w}{N}$ to the mediator.

1(c) The mediator gives each non-winning person their fair share.

Notice that Step 1 (a)–(c) is the same as Knaster’s Procedure and again may produce a surplus.

2 In addition, each person receives a proportion of the surplus.

First, some notation: $N \cdot m = b_1 + b_2 + b_3 + \ldots + b_N$.

- Person 1 receives: $\frac{b_1}{N \cdot m} (w - m)$.
- Person 2 receives: $\frac{b_2}{N \cdot m} (w - m)$.
- In general, Person $k$ receives: $\frac{b_k}{N \cdot m} (w - m)$. 
Beyond the Formula: Equitability Procedure

Let’s go beyond the formula for the Equitability Procedure by looking at the components piece-by-piece. In general, the amount given to Person $k$ in Step 2 is

$$\frac{b_k}{N \cdot m} \cdot (w - m)$$

You can work with our notation to note the following:

- $(w - m)$ is just the available surplus made from the winner paying for the resource.

- $N \cdot m = b_1 + b_2 + b_3 + \ldots + b_N$ is the sum of all bids.

- We can describe the formula above using words as:

  \[
  \frac{\text{A Person's Bid(s)}}{\text{Sum of All Bids}} \cdot (\text{Available Surplus})
  \]

This is an especially helpful view when there is MORE THAN ONE RESOURCE!
Recall the Professors who bid to for Spot #19 in line at the DMV:

<table>
<thead>
<tr>
<th></th>
<th>Math</th>
<th>Biology</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot #19</td>
<td>$60</td>
<td>$80</td>
</tr>
</tbody>
</table>

The two Professors decide to use the **Equitability Procedure**. Recall that at the end of Step 1 of Knaster’s Procedure, the Biology Professor gets the spot but pays $40 and the Math Professor gets $30. The surplus is $10.

- What is $m$?
- What is $N \cdot m$?
- Calculate $\frac{b_{\text{Math}}}{N \cdot m} (w - m)$.
- Calculate $\frac{b_{\text{Biology}}}{N \cdot m} (w - m)$.
- What is the compensation for each Professor?
Our Average Young Ladies from before decide to use the **Equitability Procedure** for disputed resource.

<table>
<thead>
<tr>
<th>The Burn Book</th>
<th>Regina</th>
<th>Gretchen</th>
<th>Karen</th>
</tr>
</thead>
<tbody>
<tr>
<td>$300</td>
<td>$210</td>
<td>$240</td>
<td></td>
</tr>
</tbody>
</table>

From Step 1 of Knaster’s, Regina gets the book but pays $300 − $\frac{300}{3} = 200$. Gretchen receives $70$ and Karen receives $80$. The surplus is $50$.

- What is $N \cdot m$?
- Calculate $\frac{b_{\text{Regina}}}{N \cdot m} (w - m)$.
- Calculate $\frac{b_{\text{Gretchen}}}{N \cdot m} (w - m)$.
- Calculate $\frac{b_{\text{Karen}}}{N \cdot m} (w - m)$.
- Write sentences describing what each Young Lady receives.
Recall that Lindsay and Tobias are fighting over **Diamond Skin Cream** and a **Poofy Blouse/Shirt**. Here are their bids:

<table>
<thead>
<tr>
<th></th>
<th>Lindsay</th>
<th>Tobias</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Diamond Skin Cream</strong></td>
<td>$200</td>
<td>$260</td>
</tr>
<tr>
<td><strong>Poofy Blouse/Shirt</strong></td>
<td>$100</td>
<td>$80</td>
</tr>
</tbody>
</table>

They decide to use the **Equitability Procedure** to divide the goods. Recall that after Step 1 of Knaster’s Procedure, Tobias gets the Skin Cream and pays $90 while Lindsay gets the Blouse and receives $50. The surplus is $40.

- Calculate \( \frac{b_{Tobias}}{N \cdot m} (w - m) \).
- Calculate \( \frac{b_{Lindsay}}{N \cdot m} (w - m) \).
- Find \( x_{Tobias} \) and \( x_{Lindsay} \).
- What are \( \frac{b_{Tobias}}{x_{Tobias}} \) and \( \frac{b_{Lindsay}}{x_{Lindsay}} \)?
Miley and Liam 4

Recall that Miley and Liam have broken up and the chart below represents what they each would bid on shared items (all in thousands of dollars):

<table>
<thead>
<tr>
<th>Item</th>
<th>Miley</th>
<th>Liam</th>
</tr>
</thead>
<tbody>
<tr>
<td>Engagement Ring</td>
<td>$400</td>
<td>$250</td>
</tr>
<tr>
<td>Warehouse of Teddy Bear Costumes</td>
<td>$100</td>
<td>$60</td>
</tr>
<tr>
<td>Basic Human Dignity</td>
<td>$30</td>
<td>$200</td>
</tr>
</tbody>
</table>

They decide to use the **Equitability Procedure** to divide the goods.

- Determine the TOTAL SURPLUS at the end of Step 1.
- Calculate $\frac{b_{Miley}}{N \cdot m} (w - m)$ and $\frac{b_{Liam}}{N \cdot m} (w - m)$.
- Find the compensation amounts for Miley and Liam. Describe what each person receives using a sentence.
- Compare $\frac{b_{Miley}}{x_{Miley}}$ with $\frac{b_{Liam}}{x_{Liam}}$. 
1: Person 1 and Person 2 each have 100 points to use in making their bids. The first step is to assign points to the resources that are to be divided. The higher number of points given, the more a person wants that particular resource.

2: At first, the Highest Bidder for each resource gets that resource. For each item/resource received, the points used to get that resource must be counted.

**Tie Breaker:** If there is a tie (same number of points) for a resource, it will go to the person who has used the fewest points so far.

The goal is for each person to receive resources *USING THE SAME NUMBER OF POINTS.*
The next step is to “Adjust” the winner of some resource(s) so that an equal number of points has been spent by both people.

3: If points used are not equal then resources are transferred from the person with the HIGHER number of points to the person with the LOWER number of points. Start by transferring items with the smallest Point Ratio (defined on the next slide).

4: If transferring an item moves too many points (causing the HIGHER and LOWER people to switch) then only a portion of the last item is transferred over. A Transfer Equation must be set up and solved to determine the amount shared. (Detailed descriptions of these equations are given later in the slides.)

The Adjusted Winner Method was first described in 1995.
Related Idea: Point Ratio

Definition
For the Adjusted Winner Method, the **Point Ratio** for each resource is the fraction

\[
\frac{\text{Points used by Winner of Resource}}{\text{Points used by Non-Winner of Resource}}
\]

Example (Point Ratio)

Siblings Fred and George on the following:

<table>
<thead>
<tr>
<th>Resource</th>
<th>Fred</th>
<th>George</th>
<th>Points Used</th>
<th>Point Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skiving Snackboxes</td>
<td>15</td>
<td>30</td>
<td>60</td>
<td>30/15 = 2</td>
</tr>
<tr>
<td>Extendable Ears</td>
<td>60</td>
<td>10</td>
<td>70</td>
<td>60/10 = 6</td>
</tr>
<tr>
<td>Pygmy Puffs</td>
<td>25</td>
<td>40</td>
<td></td>
<td>40/25 = 1.6</td>
</tr>
<tr>
<td>Points Used</td>
<td>60</td>
<td>70</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Related Idea: Transfer Equation

Definition

The **Transfer Equation** for a shared item is given by

Person 1 non-shared points $+ x \cdot P1$’s points for shared item $=$

Person 2 non-shared points $+ (1 - x) \cdot P2$’s points for shared item.

Here $x$ represents the amount of the shared resource owned by P1.

Example (Transfer Equation)

In the previous example, Fred will initially be awarded the Extendable Ears while George will initially receive the Skiving Snackboxes and the Pygmy Puggs. The Pygmy Puffs have the lowest *Point Ratio*, so they must be shared. Fred has 60 (from Extendable Ears) non-shared points while George has 30 (from the Skiving Snackboxes). The transfer equation is

$$60 + x \cdot 25 = 30 + (1 - x) \cdot 40.$$
Homework Assignments

1. HW 12 - Compensation 2 - due Thurs 11/7 - 6:00 AM

2. Decide who you would like to work with on the written project.

3. Begin studying for Quiz 3 - Fri 11/8