New Topic: Graph Theory

Our last topic of the course is the mathematics of connections, associations, and relationships.

Definition

A **Graph** is a set of points called **Vertices** (singular **Vertex**) and lines called **Edges** that connect **some** of the vertices.

Graphs are used to understand links between objects and people. The study of the mathematical properties of graphs is called **Graph Theory**.

Example

Below is a graph with 7 vertices and 10 edges.

```
Notice there is no vertex at the overlap between two edges. Vertices will/must always be clearly distinguished!
```
Examples in Graph Theory

Example (Graphs)
All of the following are graphs:

1. 
2. 
3. 

Multi-edges, loops, and two or more pieces are all allowed.

Example (Not Graphs)
None of the following are graphs:

1. 
2. 
3. 

Scenarios with no vertices, edges without ending vertices, and infinite vertices will not be allowed here.
A group of college friends discuss which classes they will be taking next semester. They make a chart of their names and all their classes below. In the chart, a “✓” means that person listed in that row is taking the class listed in that column.

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- Make a graph where the vertices are “People” and edges connect “Possible Study Partners (for any class)”.
- How many vertices are in your graph? How many edges?
A group of college friends discuss which classes they will be taking next semester. They make a chart of their names and all their classes below. In the chart, a “✓” means that person listed in that row is taking the class listed in that column.

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- Make a new graph with an edge for every class that could be studied together.
- How many vertices are in your graph? How many edges?
Some people decide to change up their schedules.

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- Make a graph with an edge for every class that could be studied together.
- How many vertices are in your graph? How many edges?
Descriptions of Graphs

REMEMBER, graphs are supposed to indicate connections between things. There are many different ways to describe these connections. Consider the graph below:

Each vertex is labeled with a capital letter. This is enough to describe the graph in its entirety.

- The **Vertex Set** is just a collection given by the labels we put on the vertices. In the example above, the vertex set is \{A, B, C, D\}.

- The **Edge Set** is the collection of the edges, described using the endpoint vertices of each edge. In the example above, the edge set is \{AB, AC, AD, BC, CD\}.
There are a few more vocabulary words that often get used with graphs.

Definition

- We usually use the letter $v$ to indicate the number of vertices of a graph. This is just a count of things that appear in the vertex set. Sometimes we call this number the **Order** of the graph.

- We usually use the letter $e$ to indicate the number of edges of a graph. This is just a count of things that appear in the edge set.

- A graph is **Connected** if it is all one piece. Stated in another way, from any starting vertex we can trace along the edges of the graph to any other vertex **WITHOUT ANY JUMPS!**
In the graph above, we have \( v = 4 \). In other words, the order of the graph is 4. Notice this is the same as the number of things in the vertex set \( \{A, B, C, D\} \).

This graph also have 7 edges. So \( e = 7 \). This value is the same as the number of things in the edge set

\[
\{AB, AC, AD, BC, BD, CD, CD\}.
\]

This graph is all one piece, so it is \textit{connected}. 
A few more friends (Erica, Fiona, and Greg) want to compare upcoming class schedules with everyone before. The friends decide that it is too much trouble to make a new chart. Instead, they list connections whenever two people have at least one class in common. They get the following list:

\[
\{AB, AC, AD, AE, BD, BE, BG, CD, CE, DE, EG\}
\]

- Draw a graph that represents the classes in common between friends A through G.
- What is the order of your graph?
- Is your graph connected?
Definition

- A graph is **Disconnected/Unconnected** if it has more than one piece. In other words, this means it is NOT ONE PIECE.

- A graph is **simple** if it has no loops nor multi-edges.

- The connected pieces of a graph are called the **Components**. We will usually use the letter $c$ to describe the number of components a graph has. A connected graph has $c = 1$. Any graph with $c > 1$ must be disconnected.

This graph is disconnected with $v = 7$, $e = 8$, and $c = 2$. 
Consider the edge set below:

\[ \{AB, AC, AD, CD, EF, FG, HI, HJ, HL, LM, LN, MN\} \]

- Draw a graph with vertices A through N and edges listed above.
- How many components does your graph have?
- Is your graph the same as the graph below?
Isomorphic Graphs

Some graphs are the “same” even though they aren’t drawn in the same way.

The graphs below are all the same, because they represent identical kinds of connections.

Definition (Isomorphic Graphs)
Graphs that have the same number of edges and identical connections (but may look different) are said to be **isomorphic**.
Are the two graphs below isomorphic?

Are the two graphs below isomorphic?
Isomorphic or Not Isomorphic 2

- Draw the graph with vertices $A, B, C, D$ and edge set $\{AB, AC, AD, BC\}$

- Is your graph isomorphic to one of the graphs below?

- Can you relabel the vertices of either pictured graph to match your graph?
Draw the graph with vertices $A, B, C, D$ and edge set

$$\{AB, AC, AD, BC, BD\}$$

Is your graph isomorphic to one of the graphs below?

Can you relabel the vertices of either pictured graph to match your graph?
Determining If Two Graphs Are Isomorphic 1

Given two graphs, it is often really hard to tell if they ARE isomorphic, but usually easier to see if they ARE NOT isomorphic. Here is our first idea to help tell if two graphs are isomorphic.

Theorem (Isomorphic Graphs Theorem 1)

Suppose we have two graphs. In the first graph there are $v_1$ vertices and $e_1$ edges. In the second graph there are $v_2$ vertices and $e_2$ edges. Then in order for the two graphs to be isomorphic we must have:

- $v_1 = v_2$
- $e_1 = e_2$

In words, isomorphic graphs must have the same number of vertices and edges.

It is important to note that just having $v_1 = v_2$ and $e_1 = e_2$ is NOT a guarantee that two graphs will be isomorphic.
Consider the following graphs below:

What is a reason(s) for why these graphs could be isomorphic?

What is a reason(s) for why these graphs could NOT be isomorphic?
Consider the following graphs below:

What is a reason(s) for why these graphs could be isomorphic?

What is a reason(s) for why these graphs could NOT be isomorphic?
Consider the following graphs below:

What is a reason(s) for why these graphs could be isomorphic?

What is a reason(s) for why these graphs could *NOT* be isomorphic?
Degree of a Vertex

Definition

The **degree of a vertex** is the number of edges attached to that vertex. In a simple graph, a vertex’s degree is strictly less than the order of the graph.

Example

We can use the idea of degree of a vertex to help us better understand when two graphs might be isomorphic.
Theorem (Isomorphic Graphs Theorem 2)

Suppose we have two graphs where each graph has the same number of vertices, \( v_1 = v_2 = n \). Write the degrees of each vertex (with repeats) in ascending order for Graph 1. This gives a list of numbers that we can represent generally as \( d_1, d_2, d_3, \ldots, d_n \).

If the two graphs are isomorphic, then when listing the degrees of Graph 2 in ascending order, we get the exact same list as above.

In short, **ISOMORPHIC GRAPHS HAVE THE SAME DEGREE LISTS.**

More useful though, **IF THE DEGREE LISTS ARE DIFFERENT, THE TWO GRAPHS ARE NOT ISOMORPHIC.**

There are more sophisticated ways to determine if two graphs are isomorphic, but generally this is a VERY HARD question to resolve.
Homework Assignments

1. read pp. 117-121 on “Graph Theory”

2. HW 15 - Graphs 1 - due Fri 11/22 - 11:59 PM