Graph Theory - Day 2: Isomorphism & Planarity

MA 111: Intro to Contemporary Math

November 22, 2013
Graph Theory Basics

Definition
A Graph is a set of points called Vertices (singular Vertex) and lines called Edges that connect some of the vertices.

Example

1. 
2. 
3. 

Multi-edges, loops, and two or more pieces are all allowed.

Definition
A simple graph has no multi-edges nor loops.
Terminology

- The **Vertex Set** is the collection of vertex labels.

- \( v = \) size of the vertex set; sometimes called the **order** of the graph

- The **Edge Set** is the collection of the edges, described using the endpoint vertices of each edge.

- \( e = \) size of the edge set

- A graph is **Connected** if it is all one piece and **Disconnected** if it has two or more pieces.

- The connected pieces of a graph are called the **Components**.

- \( c = \) the number of components in a given graph. A connected graph has \( c = 1 \), and any graph with \( c > 1 \) must be disconnected.
Isomorphic Graphs

Definition (Isomorphic Graphs)
Graphs that have the same fundamental shape and identical connections (but may be drawn differently) are said to be isomorphic.

Theorem (Isomorphic Graphs Theorem 1)
Suppose we have two graphs. The first has $v_1$ vertices and $e_1$ edges, and the second has $v_2$ vertices and $e_2$ edges. If the graphs are isomorphic, then we must have:

$$v_1 = v_2 \text{ and } e_1 = e_2$$

So, isomorphic graphs must have the same number of vertices and edges. If their $v$’s or $e$’s are different, then the two graphs are NOT isomorphic.

It is important to note that just having $v_1 = v_2$ and $e_1 = e_2$ is NOT a guarantee that two graphs will be isomorphic.
Degree of a Vertex

Definition
The **degree of a vertex** is the number of edges attached to that vertex.

Note: If we have a simple graph, then we must have that a vertex’s degree is strictly less than the order of the graph.

We can use the idea of degree of a vertex to help us better understand when two graphs might be isomorphic.
Using Vertex Degrees 1

- Make a graph with 5 vertices labeled with the letters A, B, C, D, and E.
- Include 8 edges connecting the vertices. You decide which connections to make!
- Is your graph isomorphic to this one?
Using Vertex Degrees 2

- Make another graph with 5 vertices labeled with the letters V, W, X, Y, and Z.
- Connect the vertices so that the degrees are 2, 3, 3, 4, 4
- Is your graph isomorphic to this one?
Using Vertex Degrees 3

Are the two graphs below isomorphic?

Are the two graphs below isomorphic?
Theorem (Isomorphic Graphs Theorem 2)

Suppose we have two graphs with the same order, i.e. \( v_1 = v_2 = n \). Write the degrees of each vertex (with repeats) in ascending order for Graph 1. This gives a list of numbers that we can represent generally as \( d_1, d_2, d_3, \ldots, d_n \). Do the same for Graph 2.

If the two graphs are isomorphic, then both lists must be identical.

In short, **ISOMORPHIC GRAPHS HAVE THE SAME DEGREE LISTS**.

More useful though, **IF THE DEGREE LISTS ARE DIFFERENT, THE TWO GRAPHS ARE NOT ISOMORPHIC**.

There are more sophisticated ways to determine if two graphs are isomorphic, but generally this is a VERY HARD question to resolve.
Graph Isomorphism

The “Isomorphic Graphs Theorems 1 & 2” are mostly useful for showing that two graphs are **NOT** isomorphic.

**Definition (Graph Isomorphism)**

If two graphs are isomorphic, then there is a **Graph Isomorphism** that describes how they are the same. In practice this is:

- A relabeling of the vertices of Graph 1 so that each corresponds to the “same” vertex of Graph 2;

- This relabeling is done so that any edge of Graph 1 has a corresponding edge of Graph 2 under the new labels.

To determine a graph isomorphism, a really good place to start is to find the degrees of the vertices of **BOTH** graphs.
Graph Isomorphism Procedure: Step 1

Example

Step 1: List the degrees, ascending order, of both graphs:

Left Graph

\[
\begin{align*}
3 & \quad A, \\
3 & \quad E, \\
3 & \quad F, \\
4 & \quad B, \\
4 & \quad C, \\
5 & \quad D
\end{align*}
\]

Right Graph

\[
\begin{align*}
3 & \quad W, \\
3 & \quad Y, \\
3 & \quad Z, \\
4 & \quad U, \\
4 & \quad X, \\
5 & \quad V
\end{align*}
\]
Graph Isomorphism Procedure: Step 2

Example

![Graphs]

- Step 2: Understand the connections of the vertices:
  - In the Left Graph, vertices A, E, and F all connect with B, C, and D. These are their only connections.
  - In the Right Graph, vertices W, Y, and Z all connect with U, V, and X. These are their only connections.
  - In the both graphs, Vertices of degree 4 connect with the one Vertex of Degree 5.
Step 3: Define the isomorphism:

- Because of the types of connections, any vertex of degree 3 in the Left Graph is like any vertex of degree 3 in the Right Graph. The same is true for the vertices of degree 4.
- $A \leftrightarrow W$, $E \leftrightarrow Y$, $F \leftrightarrow Z$, $B \leftrightarrow U$, $C \leftrightarrow X$, $D \leftrightarrow V$.
- We can check by redrawing the Right Graph with vertices in the positions given by the isomorphism.
Sum of Degrees of Vertices Theorem

Theorem (Sum of Degrees of Vertices Theorem)

Suppose a graph has $n$ vertices with degrees $d_1, d_2, d_3, \ldots, d_n$. Add together all degrees to get a new number $d_1 + d_2 + d_3 + \ldots + d_n = D_v$. Then $D_v = 2e$.

In words, for any graph the sum of the degrees of the vertices equals twice the number of edges.

Stated in a slightly different way, $D_v = 2e$ says that $D_v$ is always an even number.

Example

It is impossible to make a graph with $n = 6$ where the vertices have degrees 1, 2, 2, 3, 3, 4. This is because the sum of the degrees $D_v$ is

$$D_v = 1 + 2 + 2 + 3 + 3 + 4 = 15$$

$D_v$ is always an even number but 15 is odd!
Using the Sum of the Degrees of Vertices Formula 1

Consider the following scenarios:

- A graph has 4 vertices with degrees 0, 0, 0, and 0. What does this graph look like?

- A graph has 1 vertex with degree 2. What does this graph look like?

- A graph has 4 vertices with degrees 2, 3, 3, and 4. How many edges are there?

- A graph has 4 vertices with degrees 2, 2, 2, and 4. Can you say what this graph looks like?
Using the Sum of the Degrees of Vertices Formula 2

Consider the following scenarios:

- Is it possible to have a graph with vertices of degrees: 1 and 1?

- Is it possible to have a graph with vertices of degrees: 1 and 2?

- Is it possible to have a graph with vertices of degrees: 1, 1, 2, 3?

- Is it possible to have a graph with vertices of degrees: 1, 1, 2, 3, 3?
Planar Graphs

Definition
A graph is **Planar** if it can be drawn in such a way that its edges do not cross.

To determine if a graph is planar we have to consider isomorphic versions of the graph.

**Example (Using Isomorphisms to Make Planar)**
The graph on the left is definitely planar (edges do not cross). Even though the graph to the right does not appear to be planar, *it is because it is isomorphic to the graph to the left!*
Definition
In any planar graph, drawn with no intersections, the edges divide the planes into different regions.

The regions enclosed by the planar graph are called **interior faces** of the graph.

The region surrounding the planar graph is called the **exterior (or infinite) face** of the graph.

When we say **faces** of the graph we mean **BOTH** the interior AND the exterior faces. We usually denote the number of faces of a planar graph by $f$.

**BEFORE YOU COUNT FACES, IT IS VERY IMPORTANT TO FIRST DRAW A PLANAR GRAPH SO THAT NO EDGES CROSS!**
Counting Faces of a Planar Graphs

It is really tempting to try to count the faces in the graph below:

Remember, the idea of faces only applies to Planar Graphs drawn with no edge crossings. Instead we need to redraw the graph to get an isomorphic version:

In the bottom version it is easy to see there are 3 Interior Faces and 1 Exterior Face for a total of 4 Faces.
Here are a couple more graphs that don’t look planar at first. Can you count the faces?

<table>
<thead>
<tr>
<th></th>
<th>Left Graph (Cube)</th>
<th>Right Graph ($K_{3,2}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertices $v$</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>Edges $e$</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>Faces $f$</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

These numbers are giving more evidence of some big idea with planar graphs.
Homework Assignments

1. read pp. 117-121 on “Graph Theory”

2. HW 15 - Graphs 1 - due Fri 11/22 - 11:59 PM