Graph Theory - Day 7: Eulerian Circuits & Trails

MA 111: Intro to Contemporary Math

December 9, 2013
Graph Modeling: Deer John

John the Deer (pictured) is friends with the bear from the previous question. He learns graph theory from the bear and uses it to understand the regions where he (John the Deer) likes to hang.

- Make a graph that uses regions and borders for vertices and edges.
Can John the deer make a round-trip, crossing every border, without having to cross any border twice?

Can John the deer make ANY trip, crossing every border, without having to cross any border twice?
**Circuits and Trails**

**Definition**

- A **trail** of a graph is a connected route across distinct edges that begins at some vertex and ends at a *different* vertex. We may repeat vertices, but not edges.

- A **circuit** of a graph is trail that begins and ends at the same vertex.

Pictured is a trail from A to B (or B to A). One more edge (A to B) will make a circuit.
Eulerian Graphs

Definition
A graph circuit that covers every edge *EXACTLY ONCE* is called an **Euler Circuit**.

A graph is said to be **Eulerian** if there is an **Euler circuit**.

Example (Euler Circuit)
Semi-Eulerian Graphs

Definition
A graph trail that covers every edge EXACTLY once is called an Euler trail.

A graph is said to be Semi-Eulerian if there is an Euler trail.

Example (Euler Trail)
Can you find a *circuit* that hits every edge without repeating any edges?

What are the degrees of the vertices of this graph?
Can you find a circuit that hits every edge without repeating any edges?

What are the degrees of the vertices of this graph?
Travel Every Edge 3

- In the graphs above, can you find a circuit that hits every edge without repeating any edges?
- Find the degrees of the vertices of the graphs above!
- Make a conjecture (or guess) about how to tell when you can get circuits to travel over every edge.
Euler’s Theorems 1: Euler Circuits

We are ready for our first **BIG** Theorem in Graph Theory. These are the results that started Graph Theory about 300 years ago!

**Theorem (Euler Circuit Theorem)**

*A connected graph has an Eulerian Circuit exactly when every vertex has even degree.*

**Example (Degrees and Euler Circuits)**

![Graph with degrees and Eulerian circuit](image)
Euler Circuits 1

- Find the degrees of the vertices of the graphs above!
- Which of the graphs has an Euler Circuit?
- Where can your Euler Circuit begin and end?
Find the degrees of the vertices of the graphs above!

Which of the graphs has an Euler Circuit?

Where can your Euler Circuit begin and end?
Can you find a *trail* (not a round trip!) that hits every edge without repeating any edges?

What are the degrees of the vertices of this graph?
Travel Every Edge 6

- Can you find a *trail* (not a round trip!) that hits every edge without repeating any edges?
- What are the degrees of the vertices of this graph?
Travel Every Edge 7

In the graphs above, can you find a trail that hits every edge without repeating any edges?

Find the degrees of the vertices of the graphs above!

Make a conjecture (or guess) about how to tell when you can get a trail to travel over every edge.
Euler’s Theorems 2: Euler Trails

Theorem (Euler Trail Theorem)
A connected graph contains an Euler Trail exactly when all but two vertices have even degree.

The Euler Trail will begin and end at the vertices with odd degree.

Example (Degrees and Euler Paths)

WARNING: Euler’s Theorems only tell us when Euler Circuits or Euler Trails EXIST. They do not tell us how to find them.
Euler Trails 1

Find the degrees of the vertices of the graphs above!

Which of the graphs has an Euler Trail?

Where must your Euler Trail(s) begin and end?
Find the degrees of the vertices of the graphs above!

Which of the graphs has an Euler Trail?

Where must your Euler Trail(s) begin and end?
Euler’s House

Euler likes to think about his house, whose blueprint is below:

▶ Can Euler travel through every doorway exactly once to make an Euler Circuit?
▶ Can Euler travel through every doorway exactly once to make an Euler Path?
Homework Assignments

1. HW 16 - Graphs 2 (to be posted later today) - due Thurs 12/12, 6:00 AM

2. Read pp. 127-129 on “Euler Circuits”

3. Read pp. 131-132 on “Eulerization and the Chinese Postman Problem”