MA 114: Calculus II
Exam 2 Review

§10.6 Power Series
definition of a power series; how to find the radius of convergence and the interval of convergence (including endpoints); power series for \( \frac{1}{1-x} \), \( e^x \) and \(-\ln(x+1)\); term-by-term differentiation and integration

§8.4 Taylor Polynomials
definition of a Taylor polynomial and how to find one

§10.7 Taylor Series
definition of Taylor and Maclaurin series; how to find one Taylor series based on a known similar series

§6.2 Setting Up Integrals: Volume, Density, Average Value
how to set up integrals for volume based on known cross-sections; integrals for linear vs. radial density functions; how to find the average value of a function

§6.3 Volumes of Revolution
Disk vs. Washer method; how to rotate around horizontal axes and changes required to rotate around vertical axes; how to draw the region and identify radii

§6.4 The Method of Cylindrical Shells
Shell method; how to draw the region and representative rectangles; how to measure height and radius; how to rotate around both types of axes

§6.5 Work and Energy
Hooke's Law and integrals for springs; finding integral expressions for building structures and emptying water tanks; differences between building and emptying

§7.2 Trigonometric Integrals
Cases 1, 2, and 3 on evaluating integrals of powers of sine and cosine; integrals of \( \tan x \) and \( \sec x \); Cases 1, 2, and 3 on evaluating integrals of powers of tangent and secant

§7.3 Trigonometric Substitutions
identifying Cases 1, 2, and 3 of square root integrals; making the correct substitution to transform into a trigonometric integral; converting trigonometric solution back into terms of \( x \)
1. Find the center, radius of convergence, and interval of convergence (including endpoints) for
\[ \sum_{n=0}^{\infty} \frac{(-1)^n n}{5^n} (x - 2)^n. \]

2. Find the third Maclaurin polynomial for \( f(x) = x^2e^x \), and then find its full Maclaurin series.

3. Find the Taylor series for \( f(x) = \frac{1}{x} \) centered at \( x = 1 \).

4. Odzala National Park in the Republic of the Congo has a high density of gorillas. Suppose that the radial population density is \( \rho(r) = 52(1 + r^2)^{-2} \) gorillas per square kilometer, where \( r \) is the distance from a grassy clearing with a source of water. Calculate the number of gorillas within a 5-km radius of the clearing.

5. Find the average value of \( f(x) = 2x^3 - 6x^2 \) on the interval \([-1, 3]\).

6. Consider the region bounded by \( x = \sqrt{2} y, x = 0 \) and \( y = 9 \). Set up integrals required to calculate the volume of the solid generated by rotating around:
   a.) \( y \)-axis using Disk Method
   b.) \( y \)-axis using Shell Method
   c.) \( x = -3 \) using Washer Method
   d.) \( x = -3 \) using Shell Method

7. Consider a cylindrical tank of height 6 meters and radius 1.5 meters. Suppose that the tank is half-filled with a liquid of density 350 kg/m\(^3\). Calculate the work required to pump the water out of the tank through a hole at the top.

8. Suppose we want to build a rectangular tower of height 12 meters and square base of side length 5 meters. Calculate the work required to build this tower with material of density 1200 kg/m\(^3\).

9. Evaluate the following integrals:
   a.) \( \int \cos^3(\pi x) \sin(\pi x) \, dx \)
   b.) \( \int \tan^3(x) \sec^2(x) \, dx \)
   c.) \( \int x^3 \sqrt{9 - x^2} \, dx \)
   d.) \( \int \frac{x^2}{(25x^2 + 25)^{3/2}} \, dx \)
1) \[ \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} (x-2)^n \] center is 2
(match \((x-c)^n\) to \((x-2)^n\))

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}}{5^{n+1}} (x-2)^{n+1}}{\frac{(-1)^n}{5^n} (x-2)^n} \right| = \lim_{n \to \infty} \left| \frac{(-1)(n+1)}{n} (x-2) \right| = \lim_{n \to \infty} \frac{|x-2|}{5} \]

\[ = \frac{|x-2|}{5} (\lim_{n \to \infty} \frac{n+1}{n}) = \frac{|x-2|}{5} < 1 \Rightarrow |x-2| < 5 \]

So, radius of convergence is \( R = 5 \).

Interval of absolute convergence \((2-5, 2+5)\)
\((-3, 7)\).

Check endpoints:
1) \( x = -3 \):
\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} (-3-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} (-5)^n \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{5^n} = \sum_{n=0}^{\infty} n, \text{ divergent!} \]

2) \( x = 7 \):
\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} (7-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n}{5^n} (5)^n \]
\[ = \sum_{n=0}^{\infty} (-1)^n n, \text{ also divergent.} \]

So, \( I = (-3, 7) \) (endpoints are not included)
2.) \( f(x) = x^2 e^x \)
\[ f'(x) = 2x e^x + x^2 e^x = (2x + x^2) e^x \]
\[ f''(x) = (2 + 2x) e^x + (2x + x^2) e^x = (2 + 4x + x^2) e^x \]
\[ f'''(x) = (4 + 2x) e^x + (2 + 4x + x^2) e^x = (4 + 6x + x^2) e^x \]
\[ f''''(0) = (0 + 0 + 0) e^0 = 0 \]
\[ f''(0) = (2 + 0 + 0) e^0 = 2 \]
\[ f'''(0) = (6 + 0 + 0) e^0 = 6 \]

Maclaurin \( \Rightarrow c=0 \)

\[ T_3(x) = 0 + \frac{0}{1!} (x-0) + \frac{2}{2!} (x-0)^2 + \frac{6}{3!} (x-0)^3 \]
\[ = 0 + 0 + x^2 + x^3 \]
So, the 3rd Maclaurin polynomial is \( x^2 + x^3 \).

Maclaurin Series for \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \).

So, Maclaurin Series for \( x^2 e^x = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} \) \( = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} \).

3.) \( f(x) = \frac{1}{x} = x^{-1} \) So, the Taylor Series is
\[ f'(x) = -x^{-2} \]
\[ f''(x) = 2x^{-3} \]
\[ f'''(x) = -6x^{-4} \]
\[ f^{(4)}(x) = 24x^{-5} \]
\[ f^{(n)}(x) = (-1)^n n! x^{-1-n} \]
\[ f^{(n)}(1) = (-1)^n n! (1)^{-1-n} \]
\[ = (-1)^n n! \]

Also, \( \frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \)

\( = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \).
4.) \( \rho(r) = 52(1 + r^2)^{-2} \)

\[
M = 2\pi \int_0^5 r \left(52(1 + r^2)^{-2}\right)dr
\]

\[
= 2\pi \int_1^{26} 26u^{-2} du
\]

\[
= 52\pi \left( -u^{-1} \right) \bigg|_1^{26} = (52\pi) \left[ -\frac{1}{26} + 1 \right] = 50\pi \text{ gorillas}
\]

5.) \( f_{avg} = \frac{1}{3-(-1)} \int_{-1}^3 (2x^3 - 6x^2) \, dx \)

\[
= \frac{1}{4} \left[ \frac{1}{2}x^4 - 2x^3 \right] \bigg|_{-1}^3 = \frac{1}{4} \left[ \left( \frac{81}{2} - 54 \right) - \left( -\frac{1}{2} + 2 \right) \right]
\]

\[
= \frac{1}{4} (-16) = -4
\]

6.)

\[ x = \sqrt{2}y \]

\[ x = 0 \]

\[ y = 9 \]

\[ y = \frac{x}{\sqrt{2}} \]

\[ y = q \]

\[ x = 0 \]

\[ (0, q) \]

\[ (\sqrt{2}q, q) \]

\[ (0, 0) \]

\[ (\sqrt{2}(0, 9)) \]

\[ (\sqrt{2}(9, 0)) \]

a.) radius to \((x, \sqrt{2}y)\) from \(y\)-axis is \(x = \sqrt{2}y\)

\[
V = \pi \int_0^9 (\sqrt{2}y)^2 \, dy = \pi \int_0^9 2y^2 \, dy
\]

b.) radius: \(x\)

height: \(9 - \frac{y}{\sqrt{2}} = 9 - \frac{x}{\sqrt{2}}\)

\[
V = 2\pi \int_0^{\sqrt{2}(9)} x \left( 9 - \frac{x}{\sqrt{2}} \right) \, dx
\]
(c.) \[ R = 3 + x = 3 + \sqrt{2y} \]
\[ r = 3 \]
\[ V = \pi \int_{-3}^{9} (3 + \sqrt{2y})^2 - 3^2 \, dy \]

(d.) radius = \(3 + x\)
height = \(9 - y = 9 - \frac{x}{\sqrt{2}}\)
\[ V = 2\pi \int_{0}^{\sqrt{2}(9)} (3 + x)(9 - \frac{x}{\sqrt{2}}) \, dx \]

7.)

Density 350 kg/m³
Water only in height interval \([0, 3]\)

Volume of slice = \(\pi r^2 \Delta y = \pi (1.5)^2 \Delta y = \pi (2.25) \Delta y\)

Mass of slice = \(350 \times 2.25 \times \pi \Delta y = 787.5 \pi \Delta y\)
Force on slice = \(9.8 \times (787.5 \pi \Delta y) = 7717.5 \pi \Delta y\)
Work on slice = \((7717.5 \pi \Delta y)(6 - y)\)

\[ W = \int_{0}^{3} (7717.5 \pi)(6 - y) \, dy = (7717.5 \pi \left[6y - \frac{y^2}{2}\right])_{0}^{3} \]
\[ = (7717.5 \pi \left[18 - \frac{9}{2}\right] = 104186.25 \pi \text{ J} \]
8. 

\[ \text{density} \quad 1200 \text{ kg/m}^3 \]

\[ \text{volume of slice} = S^2 \Delta y = 25 \Delta y \text{ m}^3 \]

\[ \text{mass of slice} = 1200 (25 \Delta y) = 30,000 \Delta y \text{ kg} \]

\[ \text{force on slice} = 9.8 (30,000 \Delta y) = 294,000 \Delta y \text{ N} \]

\[ \text{work on slice} = (294,000 \Delta y) y \text{ N} \cdot \text{m} \]

\[ W = \int_0^{12} 294,000 y \, dy = 147,000 y^2 \bigg|_0^{12} = (147,000)(144) = 21,680,000 \text{ J} \]

9. 

a.) \[ \int \cos^3(\pi x) \sin(\pi x) \, dx \quad u = \pi x \quad du = \pi \, dx \]

\[ = \frac{1}{\pi} \int \cos^3 u \sin u \, du \quad \omega = \cos u \quad dw = -\sin u \, du \]

\[ = -\frac{1}{\pi} \int \cos^3 u (\sin u \, du) \]

\[ = -\frac{1}{\pi} \int \omega^3 \, dw = -\frac{1}{\pi} \left( \frac{\omega^4}{4} \right) + C \]

\[ = -\frac{\omega^4}{4\pi} + C = -\frac{\cos^4 u}{4\pi} + C = \boxed{\frac{-\cos^4(\pi x)}{4\pi} + C} \]

b.) \[ \int \tan^3 x \sec^2 x \, dx = \int \tan^2 x \sec x (\sec x \tan x \, dx) \]

\[ = \int (1-\sec^2 x) \sec x (\sec x \tan x \, dx) \quad u = \sec x \]

\[ du = \sec x \tan x \, dx \]

\[ = \int (1-u^2)u \, du = \int u - u^3 \, du = \frac{u^2}{2} - \frac{u^4}{4} + C \quad \text{(cont.)} \]
So, \( \int \tan^3 x \sec^2 x \, dx = \frac{1}{2} \sec^2 x - \frac{1}{4} \sec^4 x + C \)

Also, instead, let \( u = \tan x \), \( du = \sec^2 x \, dx \)
\[ \int \tan^3 x \sec^2 x \, dx = \int u^3 \, du = \frac{1}{4} u^4 + C \]
\[ = \frac{1}{4} \tan^4 x + C \]

c.) \[ \int x^3 \sqrt{9-x^2} \, dx \]
\[ x = 3 \sin \theta \]
\[ dx = 3 \cos \theta \, d\theta \]
\[ \frac{9-x^2}{3} = 3 \cos \theta \]
\[ = \int 3^5 (\sin^3 \theta \cos^2 \theta \, d\theta) = 243 \int \sin^2 \theta \cos^2 \theta (\sin \theta \, d\theta) \]
\[ = -243 \int (1 - \cos^2 \theta) \cos^2 \theta (-\sin \theta \, d\theta) \quad u = \cos \theta \]
\[ du = -\sin \theta \, d\theta \]
\[ = -243 \int (1 - u^2)u^2 \, du = -243 \int u^2 - u^4 \, du \]
\[ = -243 \left[ \frac{1}{3} u^3 - \frac{1}{5} u^5 \right] + C = -243 \left[ \frac{1}{3} \cos^3 \theta - \frac{1}{5} \cos^5 \theta \right] + C \]
\[ = 243 \left[ \frac{1}{3} \left( \frac{9-x^2}{3} \right)^3 - \frac{1}{5} \left( \frac{\sqrt{9-x^2}}{3} \right)^5 \right] + C \]
\[ = -243 \left[ \frac{(9-x^2)\sqrt{9-x^2}}{81} - \frac{(9-x^2)^2\sqrt{9-x^2}}{5(243)} \right] + C \]
\[ = \left[ -3 \frac{(9-x^2) - (9-x^2)^2}{5} \right] \sqrt{9-x^2} + C \]
\[ = \frac{-27 + 3x^2}{5} (9-x^2) \sqrt{9-x^2} + C \]
9. \[ d.) \int \frac{x^2}{(25x^2 + 25)^{3/2}} \, dx = \int \frac{x^2}{(25(x^2 + 1)^{3/2}} \, dx = \int \frac{x^2}{125(x^2 + 1)^{3/2}} \, dx \]

\[= \frac{1}{125} \int \frac{x^2}{(x^2 + 1)^{1/2}} \, dx \]

\[= \frac{1}{125} \int \frac{\tan^2 \theta}{\sec^2 \theta} \, d\theta \]

\[= \frac{1}{125} \int \frac{\sin^2 \theta}{\cos^2 \theta} \, d\theta \]

\[= \frac{1}{125} \int \frac{1 - \cos^2 \theta}{\cos^2 \theta} \, d\theta \]

\[= \frac{1}{125} \int \sec \theta - \cos \theta \, d\theta \]

\[= \frac{1}{125} \left[ \ln |\sec \theta + \tan \theta| - \sin \theta \right] + C \]

\[= \frac{1}{125} \left[ \ln \sqrt{x^2 + 1} + x - \frac{x}{\sqrt{x^2 + 1}} \right] + C \sqrt{x^2 + 1} \]

\[\sin \theta = \frac{x}{\sqrt{x^2 + 1}} \]

\[x = \tan \theta\]