7. (Challenge) Suppose \( f \) is a quadratic function with \( f(0) = 1 \), and suppose that \( \int_{0}^{1} \frac{f(x)}{x^2(x+1)^3} \, dx \) is a rational function.

Find \( f'(0) \).

Since \( f(x) \) is quadratic, \( f(x) = ax^2 + bx + c \).

Since \( f(0) = 1 \), \( f(0) = a \cdot 0^2 + b \cdot 0 + c = c = 1 \), and

\[
f(x) = ax^2 + bx + 1.
\]

\[
f'(x) = 2ax + b, \quad \text{and} \quad f'(0) = 2a \cdot 0 + b = b.
\]

Goal: find \( b \).

By partial fractions we get

\[
\frac{f(x)}{x^2(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}
\]

Since \( \int \frac{A}{x} \, dx = A \ln |x| + k \) while

\[
\int \frac{C}{x+1} \, dx = C \ln |x+1| + k \text{ are not rational functions,}
\]

\[
A = C = 0.
\]

Then \( f(x) = B(x+1)^3 + D x^2(x+1) + E x^2 \)

\[
f(x) = B (x^3 + 3x^2 + 3x + 1) + D x^3 + D x^2 + E x^2
\]

\[
f(x) = (B + D)x^3 + (3B + D + E)x^2 + (3B)x + B
\]

Since \( f(0) = B = 1 \), the coefficient on the \( x \) (b above) is \( 3B = 3 \).

Thus, \( f'(0) = 3 \).