MA 114 Worksheet # 19: Differential Equations

1. Complete the following chart:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Order</th>
<th>Linear?</th>
<th>Separable? Why?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x(y')^2 = y + x )</td>
<td></td>
<td></td>
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<tr>
<td>( x^5y' = 1 )</td>
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<tr>
<td>( \sqrt{4 - x^2}y' = e^{3y} \sin x )</td>
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<tr>
<td>( y'' = (\sin x)y' )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x^2y' + e^{-y} = 0 )</td>
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</tbody>
</table>

2. Use Separation of Variables to find the general solutions to the following differential equations.

(a) \( y' + 4xy^2 = 0 \)

(b) \( \sqrt{1 - x^2}y' = xy \)

(c) \( (1 + x^2)y' = x^3y \)

3. Solve \( y' = 4y + 24 \) subject to the condition that \( y(0) = 5 \).

4. A tank has the shape of the parabola \( y = x^2 \) revolved about the \( y \)-axis. Water leaks from a hole of area \( B = 0.0005 \, m^2 \) at the bottom of the tank. Let \( y(t) \) be the water level at time \( t \). How long does it take for the tank to empty if the initial water level is \( y(0) = 1 \, m \)?
MA 114 Worksheet # 19b: \( y' = k(y - b) \) and Graphical Methods

1. Recall that Newton's Law of Cooling stipulates that the temperature \( y(t) \) of a cooling object with respect to time satisfies the differential equation

\[
y' = -k(y - T_0),
\]

where \( k \) is a constant depending on the object and \( T_0 \) is the temperature of the ambient environment. Frank's car engine runs at 210°F. On a 70°F day, he turns off the ignition and notes that five minutes later, the engine has cooled to 160°F.

(a) Find the cooling constant \( k \).
(b) When will the engine cool to 100°F?

2. Sketch the slope field of the differential equation. Then use it to sketch a solution curve that passes through the given point

(a) \( y' = y - 2x, \quad (1, 0) \)
(b) \( y' = xy - x^2, \quad (0, 1) \)

3. Show that the isoclines of \( y' = t \) are vertical lines. Sketch the slope field for \(-2 \leq t \leq 2, \quad -2 \leq y \leq 2 \) and plot the integral curves passing through \((0, 1)\) and \((0, -1)\).
1. \( x(y')^2 = y + x \)
   - Order 1
   - Not linear because \((y')^2\)
   - Not separable
   \[ (y')^2 = \frac{y + x}{x} = \frac{y}{x} + 1 \Rightarrow y' = \sqrt{x(y + 1)} \]
   Cannot write as a product \( f(x) g(y) \).

- \( x^5 y' = 1 \)
  - Order 1
  - Linear
  - Separable: \( y' = x^{-5} \)
  \[
  \frac{dy}{dx} = (x^{-5})(1) = \frac{f(x)}{g(y)}
  \]

- \( \sqrt{1-x^2} y' = e^{3y} \sin x \)
  - Order 1
  - Not linear because \( e^{3y} \)
  - Separable: \( y' = e^{3y} \frac{\sin x}{\sqrt{1-x^2}} \)

- \( y'' = (\sin x) y' \)
  - Order 2
  - Linear
  \[
  y'' = \frac{d}{dt} \frac{dy'}{dt} = (\sin x) z = (\sin x) y'
  \]
  Separable if \( \frac{dz}{dt} = (\sin x) z \), separable.

- \( x^2 y' + e^{-y} = 0 \)
  - Order 1
  - Not linear because \( e^{-y} \)
  \[
  x^2 y' = -e^{-y} \]
  \[
  y' = \frac{1}{x^2} (-e^{-y}) \]
  Separable
2. a.) \( y' + 4x^2y^2 = 0 \) \( \Rightarrow y' = -4x^2y^2 \)
\[
\int \frac{dy}{y^2} = \int -4x \, dx \quad \Rightarrow \quad -\frac{1}{y} = -2x^2 + C
\]
\( \Rightarrow \quad y = \frac{1}{2x^2 + C} \)

b.) \( \sqrt{1-x^2} \) \( y' = xy \) \( \Rightarrow \) \( y' = \frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}} \)
\[
\Rightarrow \int \frac{dy}{y} = \int \frac{x}{\sqrt{1-x^2}} \, dx \quad u = 1-x^2 \quad du = -2x \, dx
\]
\[
\ln |y| = \int -\frac{1}{2} \frac{1}{\sqrt{u}} \, du = -\sqrt{u} + C = -\sqrt{1-x^2} + C
\]
\( \Rightarrow \quad y = e^{-\sqrt{1-x^2} + C} = A e^{-\sqrt{1-x^2}} = y \)

c.) \( (1+x^2)y' = x^3y \) \( \Rightarrow \) \( y' = \frac{dy}{dx} = \frac{x^3}{1+x^2} \)
\[
\Rightarrow \int \frac{dy}{y} = \int \frac{x^3}{1+x^2} \, dx = \int x - \frac{x}{x^2+1} \, dx
\]
\[
\left\{ \begin{align*}
\frac{x}{x^2+1} & \quad \Rightarrow \int x \, dx - \int \frac{x}{x^2+1} \, dx \\
\frac{1}{x^3} & \quad \Rightarrow \int \frac{1}{x^3} \, dx \\
-x^3 + x & \quad \Rightarrow \int -x^3 \, dx + \int x \, dx
\end{align*} \right.
\]
\[
\Rightarrow u = x^2 + 1 \\
u = 2x \, dx
\]
c.) (cont)

\[ S_0, \int \frac{dy}{y} = \int x \, dx - \frac{1}{2} \int \frac{1}{u} \, du \]

\[ \ln |y| = \frac{1}{2} x^2 - \frac{1}{2} \ln |u| + C \]

\[ \ln |y| = \frac{1}{2} x^2 - \frac{1}{2} \ln |x^2 + 1| + C \]

\[ \Rightarrow y = A e^{\frac{x^2}{2} - \frac{\ln |x^2 + 1|}{2}} = A \frac{e^{\frac{x^2}{2}}}{e^{\frac{\ln |x^2 + 1|}{2}}} = \frac{A e^{\frac{x^2}{2}}}{\sqrt{x^2 + 1}} = y \]

3.) \( y' = 4y + 24 \implies \frac{dy}{dx} = 4(y + 6) \)

\[ y(0) = 5 \implies \int \frac{dy}{y + 6} = \int 4 \, dx \]

\[ \Rightarrow \ln |y + 6| = 4x + C \implies y + 6 = Ae^{4x} \]

\[ y = Ae^{4x} - 6 \]

\[ S = Ae^{4(0)} - 6 = A - 6 \implies A = 11 \]

\[ y = 11 e^{4x} - 6 \]
$4.)$

$$A(y) = \pi r^2$$

$r = x = \sqrt{y}$ since $y = x^2$  \[ A(y) = \pi (\sqrt{y})^2 \]

$v(y) = -4.43 \sqrt{y}$ by Torricelli

$$\frac{dy}{dx} = \frac{-4.43 \sqrt{y}(0.0005)}{\pi y}$$

$$\int \frac{y}{\sqrt{y}} \ dy = \int \left(\frac{-4.43(0.0005)}{\pi} \right) \ dx$$

$$\int \sqrt{y} \ dy = \frac{-0.002215}{\pi} \int \ dx$$

$$\frac{2}{3} y^{3/2} = \frac{-0.002215}{\pi} x + C$$

$$y^{3/2} = \frac{3}{2} \left(\frac{-0.002215}{\pi} x + C\right)$$

$$y^{3/2} = -0.0033225 x + C$$

$$y = \left(-\frac{0.0033225}{\pi} x + C\right)^{2/3}$$

$$1 = \left(-\frac{0.0033225}{\pi} (0) + C\right)^{2/3} = (C)^{2/3} \Rightarrow C = 1$$

$$0 = \left(-\frac{0.0033225}{\pi} (x) + 1\right)^{2/3} \Rightarrow$$

$$x = (-1) \left(-\frac{0.0033225}{\pi}\right) \approx 94.5 \text{ sec} = 15.76 \text{ min}$$
1) \( y(0) = 210 \) \( \frac{y}{T_0} = 70 \) 
\( y(5) = 160 \) 
\( y(t) = 70 + A e^{-kt} \) 
\( y(0) = 70 + A e^{0} = 210 \Rightarrow A = 140 \) 
\( y(5) = 70 + 140 e^{-5k} = 160 \) 
\( 90 = 140 e^{-5k} \) 
\( -5k = \ln(9/14) \Rightarrow k = -\frac{\ln(9/14)}{5} \approx 0.0884 \text{ min}^{-1} \)

b) \( 100 = 70 + 140 e^{-0.0884t} \) 
\( 30 = 140 e^{-0.0884t} \) 
\( -0.0884t = \ln(3/14) \) 
\( t = \frac{\ln(3/14)}{-0.0884} \approx 17.43 \text{ min} \)

2) a) \( y' = y - 2x \), \((1,0)\)
\( y' = 0 = y - 2x \Rightarrow y = 2x \) 
\( y' = 1 = y - 2x \Rightarrow y = 2x + 1 \) 
\( y' = 2 = y - 2x \Rightarrow y = 2x + 2 \) 
\( y' = c \Rightarrow \text{isoclines are} \ y = 2x + c \)

b) \( y' = xy - x^2 \), \((0,1)\)
\( y' = 0 = x(y-x) \) \( x = 0 \) or \( y = x \) 
\( y' = 1 = x(y-x) \) \( y-x = \frac{1}{x} \) 
\( y = x + \frac{1}{x} = x^2 + 1 \) 
\((0,1) \Rightarrow 0 \) \((2,0) \Rightarrow 0 - 4 = -4 \) 
\((1,0) \Rightarrow -1 \) \((0,0) \Rightarrow 0 - t^2 \) 
\((1,1) \Rightarrow 1 - 1 = 0 \)
3. \( y' = t \)

\[ c = t \rightarrow x = c \text{ is a vertical line} \]

(0, 1)

(0, -1)

(0, 1)

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