MA 114 Worksheet # 3: Sequences

1. Write the first four terms of the sequences with the following general terms:
   (a) \( \frac{n!}{2^n} \)
   (b) \( \frac{n}{n + 1} \)
   (c) \((-1)^{n+1}\)

2. Find a formula for the \(n\)th term of the sequence \(\left\{\frac{1}{1}, -\frac{1}{8}, \frac{1}{27}, -\frac{1}{64}, \ldots\right\}\).

3. Conceptual Understanding:
   (a) What is a sequence?
   (b) What does it mean to say that a sequence is bounded?
   (c) What does it mean to say that a sequence is defined recursively?
   (d) What does it mean to say that a sequence converges?

4. Let \(a_0 = 0\) and \(a_1 = 1\). Write out the first five terms of \(\{a_n\}\) where \(a_n\) is recursively defined as \(a_{n+1} = 3a_{n-1} + a_n^2\).

5. Suppose that a sequence \(\{a_n\}\) is bounded above and below. Does it converge? If not, produce a counterexample.

6. Show that the sequence with general term \(a_n = \frac{3n^2}{n^2 + 2}\) is increasing. Find and upper bound. Does \(\{a_n\}\) converge?

7. Use the appropriate limit laws and theorems to determine the limit of the sequence or show that it diverges.
   (a) \(a_n = 1.01^n\).
   (b) \(b_n = \frac{3n^2 + n + 1}{2n^2 - 3}\).
   (c) \(c_n = e^{1-n^2}\).
1.) \(a_n = \frac{n!}{2^n}\)

\[
\frac{1!}{2^1} = \frac{1}{2}, \quad \frac{2!}{2^2} = \frac{1 \cdot 2}{4} = \frac{1}{2}, \\
\frac{3!}{2^3} = \frac{1 \cdot 2 \cdot 3}{8} = \frac{6}{8} = \frac{3}{4}, \quad \frac{4!}{2^4} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{16} = \frac{24}{16} = \frac{3}{2},
\]

b.) \(\frac{n}{n+1} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\)

c.) \((-1)^{n+1} (-1)^2, (-1)^3, (-1)^4, (-1)^5\)

\[1, -1, 1, -1\]

2.) \(\frac{1}{1}, -\frac{1}{8}, \frac{1}{24}, \frac{1}{64}, \quad a_n = \frac{(-1)^{n+1}}{n^3} = \frac{(-1)^{n+1}}{n^3}\)

3.) See notes.

4.) \(a_0 = 0, \quad a_1 = 1, \quad a_{n+1} = 3a_n - a_{n-1} + a_n^2\)

\[a_2 = 3a_1 + a_1^2 = 3(1) + 1^2 = 1\]

\[a_3 = 3a_2 + a_2^2 = 3(1) + 1^2 = 4\]

\[a_4 = 3a_3 + a_3^2 = 3(4) + 4^2 = 19\]

\[a_5 = 3a_4 + a_4^2 = 3(4) + 19^2 = 373\]

5.) \(m \leq a_n \leq M \quad \text{for all } n.\)

Does not necessarily converge.

\(\text{ex. } a_n = \sin(n)\)

\[-1 \leq \sin(n) \leq 1 \quad \text{but } \quad \lim_{x \to \infty} \sin(x) \text{ DNE...}\]

6.) \(a_n = \frac{3n^3}{n^2+2} = f(n) \quad f'(n) = \frac{6n(n^2+2) - (3n^3)(2n)}{(n^2+2)^2}\)

Since \(f'(n) > 0, \quad \text{increasing,}\)

\[\text{when } n \text{ is positive.}\]
(e.) cont. \( \frac{3n^2}{n^2+2} \leq \frac{3n^2}{n^2} = 3. \)

Since increasing and bounded above, must converge.

In fact, \( \lim_{n \to \infty} \frac{3n^2}{n^2+2} = \lim_{n \to \infty} \frac{3}{1 + \frac{2}{n^2}} = 3 \)

7.) a.) \( a_n = 1.01^n \) is geometric with \( c = 1, \ r = 1.01 \)

Diverges.

b.) \( b_n = \frac{3n^2+n+1}{2n^2-3} = \frac{\frac{3}{n} + \frac{1}{n^2}}{2 - \frac{3}{n^2}} \)

\[ \lim_{n \to \infty} b_n = \lim_{n \to \infty} \left( \frac{\frac{3}{n} + \frac{1}{n^2}}{2 - \frac{3}{n^2}} \right) = \frac{\lim_{n \to \infty} \left(3 + \frac{1}{n} + \frac{1}{n^2}\right)}{\lim_{n \to \infty} \left(2 - \frac{3}{n^2}\right)} \]

\[ = \frac{\lim_{n \to \infty} 3 + \lim_{n \to \infty} \frac{1}{n} + \left(\lim_{n \to \infty} \frac{1}{n}\right)^2}{\lim_{n \to \infty} 2 - 3\left(\lim_{n \to \infty} \frac{1}{n}\right)^2} \]

\[ = \frac{3 + 0 + 0^2}{2 - 3(0)^2} = \frac{3}{2} \]

\[ = e \left( \frac{1}{e} \right)^n^2 \]

\[ c.) \ C_n = e^{1-n^2} \quad f(x) = e^x \quad (e^{1-n})(e^{1+n}) \]

\[ \lim_{n \to \infty} C_n = e^{1-n^2} = \lim_{n \to \infty} \frac{1}{e^{n^2-1}} = \frac{1}{e^{\lim_{n \to \infty} (n^2-1)}} \approx \frac{1}{e^\infty} = 0 \]