1. The graph of the function \( y = f(x) \) is shown below, along with the graph of the tangent line to this curve at \( x = 3 \). Determine \( f'(3) \).

Since \( f'(3) \) is merely the slope of the given tangent line, we locate two points on the tangent line, say \((0, 2)\) and \((3, 1)\), and calculate the slope between them. So, 

\[
 f'(x) = \frac{1 - 2}{3 - 0} = \frac{-1}{3}.
\]

2. Suppose that you don’t have an explicit formula for the function \( q(x) \), but you would still like to know \( q'(1) \). Suppose that you happen to know that 

\[
 q(x+h) - q(x) = \frac{2x^2h + 15\sqrt{x}h^2 + 19h}{-x^2 + 7h^3}
\]

Find \( q'(1) \).

While we don’t have a formula for \( q(x) \), we do have \( q(x+h) - q(x) \). If we divide this quantity by \( h \), we’ll get a formula for \( \frac{q(x+h) - q(x)}{h} \).

\[
 q'(x) = \lim_{h \to 0} \frac{q(x+h) - q(x)}{h} = \lim_{h \to 0} \frac{1}{h} \cdot \frac{2x^2h + 15\sqrt{x}h^2 + 19h}{-x^2 + 7h^3} = \lim_{h \to 0} \frac{2x^2 + 15\sqrt{x}h + 19}{-x^2 + 7h^3} = \frac{2x^2 + 19}{-x^2}
\]

\[
 q'(1) = \frac{2 + 19}{-1} = -21
\]
3. Suppose that \( g(1) = 2 \) and
\[
\frac{g(x + h) - g(x)}{h} = \frac{6h + h^2}{h((x + h)^2 + 4(x + h) + 1)(x^2 + 4x + 1)}.
\]
Find an equation of the tangent line to \( y = g(x) \) at \( x = 1 \).

The equation for the tangent line in question is \( y = g(1) + g'(1)(x - 1) \). We have that \( g(1) = 2 \). Now,
\[
g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = \lim_{h \to 0} \frac{6h + h^2}{h((x + h)^2 + 4(x + h) + 1)(x^2 + 4x + 1)}
\]
\[
= \lim_{h \to 0} \frac{6 + h}{((x + h)^2 + 4(x + h) + 1)(x^2 + 4x + 1)} = \frac{6}{(x^2 + 4x + 1)^2}
\]
So, \( g'(1) = \frac{6}{(1 + 4 + 1)^2} = \frac{1}{6} \). So, the equation of the tangent line is \( y = 2 + \frac{1}{6}(x - 1) \).

4. (2 pts.) Suppose \( s(t) = \sqrt{2t - 3} \).

a.) Find constants \( A, B, \) and \( C \) such that
\[
\frac{s(t + h) - s(t)}{h} = \frac{A}{\sqrt{2t + Bh + C + \sqrt{2t - 3}}}.
\]
\[
s(t + h) - s(t) = \sqrt{2(t + h) - 3} - \sqrt{2t - 3} = \frac{\sqrt{2t + 2h - 3} - \sqrt{2t - 3}}{h} = \frac{\sqrt{2t + 2h - 3} + \sqrt{2t - 3}}{h}
\]
\[
= \frac{2t + 2h - 3 - (2t - 3)}{h(\sqrt{2t + 2h - 3} + \sqrt{2t - 3})} = \frac{2h}{h(\sqrt{2t + 2h - 3} + \sqrt{2t - 3})} = \frac{2}{\sqrt{2t + 2h - 3} + \sqrt{2t - 3}}
\]
Matching coefficients gives \( A = 2, B = 2, \) and \( C = -3 \).

b.) Calculate \( s'(t) \).
\[
s'(t) = \lim_{h \to 0} \frac{s(t + h) - s(t)}{h} = \lim_{h \to 0} \frac{2}{(\sqrt{2t + 2h - 3} + \sqrt{2t - 3})} = \frac{2}{2\sqrt{2t - 3}} = \frac{1}{\sqrt{2t - 3}}
\]

c.) Find an equation of the tangent line to \( y = s(t) \) at \( t = 6 \).
\[
\text{Since } s(6) = \sqrt{2(6) - 3} = 3 \text{ and } s'(6) = \frac{1}{3} = \frac{1}{3}, \text{ the equation of the tangent line in question is }
\]
\[
y = s(6) + s'(6)(x - 6) = 3 + \frac{1}{3}(x - 6).
\]

d.) Where does the tangent line you found for part (c) cross the \( y \)-axis? the \( t \)-axis?

The tangent line crosses the \( y \)-axis when \( t = 0 \) with a \( y \)-value of \( 3 + \frac{1}{3}(0 - 6) = 3 - 2 = 1 \), i.e at the point \( (0,1) \). The tangent line crosses the \( t \)-axis when \( y = 0 \). So, to find the corresponding value of \( t \), set the tangent line equal to 0 and solve for \( t \).
\[
0 = 3 + \frac{1}{3}(t - 6) \Rightarrow -3 = \frac{1}{3}(t - 6) \Rightarrow -9 = t - 6 \Rightarrow t = -3.
\]
Thus, the tangent line crosses the \( t \)-axis at \( (-3,0) \).
5. Suppose \( k(s) = \sqrt{s + 1} \). Use the definition of the derivative to find \( k'(8) \).

\[
k'(s) = \lim_{h \to 0} \frac{k(s + h) - k(s)}{h} = \lim_{h \to 0} \frac{\sqrt{(s + h) + 1} - \sqrt{s + 1}}{h}
\]

\[
= \lim_{h \to 0} \frac{\sqrt{s + h + 1} - \sqrt{s + 1}}{h} \cdot \frac{\sqrt{s + h + 1} + \sqrt{s + 1}}{\sqrt{s + h + 1} + \sqrt{s + 1}}
\]

\[
= \lim_{h \to 0} \frac{s + h + 1 - (s + 1)}{h(\sqrt{s + h + 1} + \sqrt{s + 1})} = \lim_{h \to 0} \frac{1}{\sqrt{s + h + 1} + \sqrt{s + 1}} = \frac{1}{2\sqrt{s + 1}}
\]

So, \( k'(8) = \frac{1}{2\sqrt{8 + 1}} = \frac{1}{6} \).

6. (2 pts.) Suppose \( f(x) = \frac{6}{3x + 1} \).

a.) Find constants \( A, B, \) and \( C \) such that \( \frac{f(x + h) - f(x)}{h} = \frac{A}{(3x + 1)(Bx + Ch + 1)} \).

\[
f(x + h) - f(x) = \frac{6}{3(x + h) + 1} - \frac{6}{3x + 1} = \frac{1}{h} \left( \frac{6(3x + 1)}{(3x + 1)(3x + 3h + 1)} - \frac{6(3x + 3h + 1)}{(3x + 1)(3x + 3h + 1)} \right)
\]

\[
= \frac{1}{h} \left( \frac{18x + 6 - 18x - 18h - 6}{(3x + 1)(3x + 3h + 1)} \right) = \frac{-18}{(3x + 1)(3x + 3h + 1)}
\]

Matching coefficients gives \( A = -18, B = 3, \) and \( C = 3 \).

b.) Calculate \( f'(x) \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{-18}{(3x + 1)(3x + 3h + 1)} = \frac{-18}{(3x + 1)^2}
\]

c.) Find an equation of the tangent line to \( y = f(x) \) at \( x = 1 \).

Since \( f(1) = \frac{6}{3 + 1} = \frac{3}{2} \) and \( f'(1) = \frac{-18}{(3 + 1)^2} = -\frac{9}{8} \), the equation of the tangent line in question is

\[
y = f(1) + f'(1)(x - 1) = \frac{3}{2} - \frac{9}{8}(x - 1).
\]

d.) Where does the tangent line you found for part (c) cross the \( y \)-axis? the \( x \)-axis?

The tangent line crosses the \( y \)-axis when \( x = 0 \) with a \( y \)-value of \( \frac{3}{2} - \frac{9}{8}(0 - 1) = \frac{3}{2} + \frac{9}{8} = 2.625 \), i.e at the point \( (0, 2.625) \). The tangent line crosses the \( x \)-axis when \( y = 0 \). So, to find the corresponding value of \( x \), set the tangent line equal to 0 and solve for \( x \).

\[
0 = \frac{3}{2} - \frac{9}{8}(x - 1) \Rightarrow -\frac{3}{2} = -\frac{9}{8}(x - 1) \Rightarrow \frac{4}{3} = x - 1 \Rightarrow x = \frac{7}{3}.
\]

Thus, the tangent line crosses the \( x \)-axis at \( \left( \frac{7}{3}, 0 \right) \).
7. Suppose \( p(x) = \frac{1}{5 - 2x} \). Use the definition of the derivative to find \( p'(4) \).

\[
p'(x) = \lim_{h \to 0} \frac{p(x + h) - p(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left( \frac{5 - 2x}{(5 - 2x)(5 - 2x - 2h)} - \frac{5 - 2x - 2h}{(5 - 2x)(5 - 2x - 2h)} \right)
= \lim_{h \to 0} \frac{2}{(5 - 2x)^2}
\]

So, \( p'(4) = \frac{2}{(5 - 8)^2} = \frac{2}{9} \).

8. Suppose \( g(x) = |2x^2 - 2x - 12| \).
   a.) Graph \( y = g(x) \).

   ![Graph of y = g(x)]

   b.) Determine the value(s) of \( x \) where \( g(x) \) is not differentiable.

   An absolute value function has points of non-differentiability when its interior function is equal to 0. So, the \( x \)-coordinates of the points where \( g(x) \) is not differentiable are the roots of \( 2x^2 - 2x - 12 = 2(x - 3)(x + 2) \), i.e. \( x = 3 \) and \( x = -2 \).
9. (2 pts.) Consider the piecewise function \( f(x) = \begin{cases} |4x - 3| & x < 2 \\ x^2 + 1 & 2 \leq x \leq 6 \\ 5x + 6 & x > 6 \end{cases} \)

a.) For which value(s) of \( x \) is \( f(x) \) not continuous?

Each piece of \( f(x) \) is continuous by itself, so we need only consider the points where the function switches pieces.

\[
\begin{align*}
\lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} |4x - 3| = 5 \\
\lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} x^2 + 1 = 5 \\
\Rightarrow \lim_{x \to 2} f(x) &= 5
\end{align*}
\]

Since \( f(2) = 5 \), we have \( \lim_{x \to 2} f(x) = f(2) \), which implies \( f \) is continuous at \( x = 2 \).

\[
\begin{align*}
\lim_{x \to 6^-} f(x) &= \lim_{x \to 6^-} x^2 + 1 = 37 \\
\lim_{x \to 6^+} f(x) &= \lim_{x \to 6^+} 5x + 6 = 36 \\
\Rightarrow \lim_{x \to 6} f(x) &= \text{DNE}
\end{align*}
\]

Since \( \lim_{x \to 6^-} f(x) \) does not exist, \( f \) is not continuous at \( x = 6 \).

b.) For which value(s) of \( x \) is \( f(x) \) not differentiable?

The second and third pieces of \( f \) are differentiable on their own, and the first piece is differentiable unless \( 4x - 3 = 0 \), that is when \( x = \frac{3}{4} \). Therefore, the only possible points where \( f \) could be not differentiable are at \( x = \frac{3}{4}, x = 2, \) and \( x = 6 \).

Since the point \( x = \frac{3}{4} \) is controlled by the first piece of \( f \), we have that \( f \) is not differentiable at \( x = \frac{3}{4} \).

Observe that if \( x = \frac{3}{4} \) were controlled by either of the other pieces, then \( f \) would be differentiable there.

Since \( f \) is not continuous at \( x = 6 \), \( f \) is not differentiable at \( x = 6 \).

Even though \( f \) is continuous at \( x = 2 \), we don't know if \( f \) is differentiable there or not. To determine differentiability, consider the derivatives/slopes on either side of \( x = 2 \). First, observe that \( |4x - 3| \) has a constant derivative of \(-4\) to the left of \( x = \frac{3}{4} \) and a constant derivative of \(4\) to the right. Since \( x = 2 \) is to the right of \( x = \frac{3}{4} \), we have the following:

\[
\begin{align*}
\lim_{x \to 2^-} f'(x) &= \lim_{x \to 2^-} |4x - 3| = 4 \\
\lim_{x \to 2^+} f'(x) &= \lim_{x \to 2^+} 2x = 4 \\
\Rightarrow \lim_{x \to 2} f'(x) &= 4
\end{align*}
\]

Since the sided limits agree, we have that \( f \) is differentiable at \( x = 2 \).
10. a.) Expand \((x - 3)^4\).

\[
(x - 3)^4 = x^4 - 4x^3 \cdot 3 + 6x^2 \cdot 3^2 - 4x \cdot 3^3 + 3^4 \\
= x^4 - 12x^3 + 54x^2 - 108x + 81
\]

b.) Expand \((x + 2)^6\).

\[
(x + 2)^6 = x^6 + 6x^5 \cdot 2 + 15x^4 \cdot 2^2 + 20x^3 \cdot 2^3 + 15x^2 \cdot 2^4 + 6x \cdot 2^5 + 2^6 \\
= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64
\]

11. Find an equation of the tangent line to \(y = (2x + 4)^3\) at \(x = -1\).

Let \(f(x) = (2x + 4)^3\).

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(2(x + h) + 4)^3 - (2x + 4)^3}{h}
\]

\[
= \lim_{h \to 0} \frac{(2(x + h))^3 + 3(2(x + h))^2 + 3(2x + 4)^2 + 3 - ((2x)^3 + 3(2x)^2 + 3(2x)^4 + 4^3)}{h}
\]

\[
= \lim_{h \to 0} \frac{8(x + h)^3 + 12(4(x + h)^2) + 48(2x + h)) + 64 - 8x^3 - 8x^2 - 64(2x) - 64}{h}
\]

\[
= \lim_{h \to 0} \frac{8(x^3 + 3x^2h + 3xh^2 + h^3) + 48(x^2 + 2xh + h^2) + 96(x + h) + 64 - 8x^3 - 48x^2 - 96x - 64}{h}
\]

\[
= \lim_{h \to 0} \frac{24x^2h + 24xh^2 + 8h^3 + 96xh + 48h^2 + 96h}{h}
\]

\[
= \lim_{h \to 0} \frac{24x^2 + 24xh + 8h^2 + 96x + 48h + 96}{h}
\]

\[
= 24x^2 + 96x + 96
\]

Since \(f(-1) = (-2 + 4)^3 = 8\) and \(f'(-1) = 24 - 96 + 96 = 24\), then the equation of the tangent line in question is \(y = f(-1) + f'(-1)(x + 1) = 8 + 24(x + 1)\).

12. Use the definition of the derivative to show that, if \(f(x) = x^4 + x^2\), then \(f'(x) = 4x^3 + 2x\).

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^4 + (x + h)^2 - (x^4 + x^2)}{h}
\]

\[
= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + x^2 + 2xh + h^2 - x^4 - x^2}{h}
\]

\[
= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2xh + h^2}{h}
\]

\[
= \lim_{h \to 0} \frac{4x^3 + 6x^2h + 4xh^2}{h} + 2x + h = 4x^3 + 2x
\]