MA123, Chapter 7: Word Problems (pp. 125-153, Gootman)

**Chapter Goals:** In this Chapter we learn a general strategy on how to approach the two main types of word problems that one usually encounters in a first Calculus course:
- Max-Min problems
- Related Rates problems

**Assignments:** Assignment 16 Assignment 17

**Suggestions:** The most important skill in solving a word problem is reading comprehension. The most important attitude to have in attacking word problems is to be willing to think about what you are reading and to give up on hoping to mechanically apply a set of steps. Nevertheless, we will present some useful strategies to employ that are often helpful.

**MAX-MIN PROBLEMS**

All max-min problems ask you to find the largest or smallest value of a function on an interval. Usually, the hard part is reading the English and finding the formula for the function. Once you have found the function, then you can use the techniques from Chapter 6 to find the largest or smallest values.

**Max-min guideline:** This guideline is found on pp. 131-133 of our textbook.

1. Read the problem quickly.
2. Read the problem carefully.
3. Define your variables. If the problem is a geometry problem, draw a picture and label it.
4. Determine whether you need to find the max or the min.
   Determine exactly what needs to be maximized or minimized.
5. Write the *general* formula for what you are trying to maximize or minimize. If this formula only involves *one* variable, then skip steps 6, 7 and 8.
6. Find the relationship(s) (i.e., equation(s)) between the variables.
7. Do the algebra to solve for one variable in the equation(s) as a function of the other(s).
8. Use your formula from step 5 to rewrite the formula that you want to maximize or minimize as a function of one variable only.
9. Write down the interval over which the above variable can vary, for the particular word problem you are solving.
10. Take the derivative and find the critical points.
11. Use the techniques from Chapter 6 to find the maximum or the minimum.

**Example 1:** What is the largest possible product you can form from two non-negative numbers whose sum is 30?

Let \( x, y \) be the nonnegative numbers.

\[
\begin{align*}
\text{Given} & \quad x + y = 30, \quad x, y \geq 0. \\
\text{Want} & \quad xy \quad \text{large} \\
\text{Put} & \quad xy = x(30-x) = 30x-x^2 \\
\text{Also} & \quad x+y \geq 20, \quad x \geq 20, \quad y \geq 20 \\
& \quad 0 \leq x, \quad y \leq 30
\end{align*}
\]

\[
\Rightarrow \quad \text{Need to find max of} \quad f(x) = 30x-x^2 \quad \text{on} \quad [0, 30].
\]

\[
f'(x) = 30 - 2x = 0 \Rightarrow x = 15.
\]

Max occurs at \( x = 0, 15, \text{or} 30. \)

Check each one:

\[
\begin{align*}
f(0) &= 0, \quad f(30) = 0 \\
f(15) &= 15 \cdot 15 = 225 = \text{Max}
\end{align*}
\]
**Example 2:** Suppose the product of \(x\) and \(y\) is 26 and both \(x\) and \(y\) are positive.

What is the minimum possible sum of \(x\) and \(y\)?

Given \(x, y > 0\), \(xy = 26 \Rightarrow y = \frac{26}{x} = 26x^{-1}\)

Want \(x + y\) minimal but

\[ x + y = x + 26x^{-1} \] so want

\[ f(x) = x + 26x^{-1} \] minimal on \([0, \infty)\)

\[ f'(x) = 1 - 26x^{-2} = 0 \Rightarrow 26x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{26}} \]

But \(x > 0\). So consider only \(x = \frac{1}{\sqrt{26}}\).

Does \(x = \frac{1}{\sqrt{26}}\) give a min or min?

\[ f''(x) = \frac{52}{26x^3} \] hence

\[ f(x) \bigg|_{x = \frac{1}{\sqrt{26}}} = \frac{7}{\sqrt{26}} \]

\[ \text{Minimum at } x = \frac{1}{\sqrt{26}}. \]

**Note:** An alternative wording for Example 2 above is:

"Suppose \(y\) is inversely proportional to \(x\) and the constant of proportionality equals 26. What is the minimum sum of \(x\) and \(y\) if \(x\) and \(y\) are both positive?"

**Example 3:** A farmer builds a rectangular pen with three parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?

Has to build \(4w + 2l = 500\) feet in length.

\[ \text{Wants Area: } A = lw \text{ max.} \]

Now \(4w + 2l = 500 \Rightarrow l = 250 - 2w \)

\[ \text{Let: } w > 0 \text{ both } \Rightarrow 250 - 2w \geq 0 \Rightarrow w \leq \frac{250}{2} \]

Area \(A = lw = (250 - 2w)w = 250w - 2w^2 \) on \([0, 125]\)

\[ A' = 250 - 4w = 0 \Rightarrow w = \frac{250}{4} = \frac{125}{2} \]

Check endpoints & critical points:

\[ A(0) = A(125) = 0, \quad A\left(\frac{125}{2}\right) = \left(125 - 2 \cdot \frac{125}{2}\right) \cdot \frac{125}{2} = \frac{125^2}{2} \]

So maximizing dimensions\(l = 125\), \(w = 125/2\).

**Example 4:** A Norman window has the shape of a rectangle capped by a semicircle. What is the length of the base of a Norman window of maximum area if the perimeter of the window equals 10?

Perimeter: \(h + b + \frac{1}{2} \text{ Circumference} = 2h + 2r + \pi r \)

\[ = 2h + (2 + \pi)r \]

Given \(2h + (2 + \pi)r = 10 \Rightarrow h = 5 - \left(\frac{2 + \pi}{2}\right)r \)

Want area max, but area: \(bh + \frac{1}{2} \text{ Area circle} \)

\[ A = bh + \frac{1}{2} \pi r^2 = (2r)(5 - \frac{2 + \pi}{2}r) + \frac{1}{2} \pi r^2 \]

\( = 10r - (2 + \pi)r^2 + \frac{1}{2} \pi r^2 = 10r - (2 + \frac{\pi}{2})r^2 \)

\[ \text{Need } A = 10r - (2 + \frac{\pi}{2})r^2 \text{ max at } r \in \left[0, \frac{10}{2 + \pi}\right] \]

\[ A'(r) = 10 - 2(2 + \pi)r = 0 \Rightarrow r = \frac{10}{2 + \pi} \]

Check endpoints & critical points, max occurs at \(r \approx \frac{5}{2 + \pi} \).

\[ b = 2r, \text{ so maximizing base } \Rightarrow b = \frac{10}{2 + \pi} \]

Endpoints: \(r = 0, 5 \Rightarrow r < \frac{2 + \pi}{2} \)

So endpoints for \(r \approx \left[0, \frac{10}{2 + \pi}\right] \)
\[ \text{Example 5:} \text{ Find the area of the largest rectangle with sides parallel to the coordinate axes that can be inscribed in a quarter circle of radius 10. Assume the center of the circle is located at the origin, and one corner of the rectangle is located at the origin and the opposite corner on the quarter circle.} \\
\text{Let } A(x) = x(100-x^2)^{1/2} \text{ on } [0, 10]. \\
\text{Area} = x(100-x^2)^{1/2} \\
\text{So, want to maximize } \text{Max } \text{Area} = A(5\sqrt{2}) = (5\sqrt{2})\sqrt{100-(5\sqrt{2})^2} = 50 \text{ (Be sure to check endpoints, } x=0). \\
\text{Example 6:} \text{ Let } A \text{ be the point } (0, 1) \text{ and let } B \text{ be the point } (5, 3). \text{ Find the length of the shortest path that connects points } A \text{ and } B \text{ if the path must touch the } x \text{-axis. In other words, the path goes from point } A \text{ to somewhere (say } P \text{) on the } x \text{-axis, and then to } B. \text{ (This is the 'line of sight' path from } A \text{ to } B \text{ if the } x \text{-axis is a mirror.) See the picture for a sketch of such a path.} \\
\text{Total distance} = (x^2 + 1)^{1/2} + ((x-5)^2 + 9)^{1/2} \text{ Why?} \\
\text{Total distance} - (\frac{\sqrt{2}}{2})(x+1) - \frac{\sqrt{2}}{2}(x-5) \text{ from } A \text{ to } P \text{ and } P \text{ to } B \text{.} \\
\Rightarrow \frac{x}{\sqrt{x^2+1}} + \frac{x-5}{\sqrt{(x-5)^2+9}} = 0 \\
\Rightarrow \frac{x}{\sqrt{x^2+1}} = \frac{x-5}{\sqrt{(x-5)^2+9}} \\
\Rightarrow x^2 = (x-5)^2 + 9 \Rightarrow x = \frac{5}{4} \text{ or } \frac{5}{4} \Rightarrow x \text{ must be } > 0. \\
\text{Minimal length is} \sqrt{(\frac{5}{4})^2 + 1} + \sqrt{(\frac{5}{4} - 5)^2 + 9} \text{.} \\
\text{Observation:} \text{ The } x \text{-coordinate of the point } P' \text{ that minimizes the line of sight path from } A(0, 1) \text{ to } B(5, 3) \text{ corresponds to the } x \text{-intercept of the line } y = \frac{1}{5}x + 1 \text{ from } A \text{ to } B'(5, -3). \text{ Note that the coordinates of } P' \text{ are } (5/4, 0). \text{ Can you understand why? Perhaps, the picture on the right will convince you.}
Example 7: Find the area of the largest rectangle with one corner at the origin, the opposite corner in the first quadrant on the graph of the parabola \( f(x) = 9 - x^2 \), and sides parallel to the axes.

\[
\text{Area} = xy = x(9-x^2) = 9x - x^3
\]
on \([0, 3]\).

\[
A'(x) = 9 - 3x^2 = 0 \Rightarrow 3x^2 = 9 \Rightarrow x^2 = 3 \Rightarrow x = \pm \sqrt{3} \leq x > 0, \text{ so only take } \sqrt{3}
\]

Check and points critical points:

\[
A(0) = A(3) = 0
\]

\[
A(\sqrt{3}) = \sqrt{3} \cdot (9 - (\sqrt{3})^2) = 6\sqrt{3}
\]

Max

Example 8: Find the point \( P \) in the first quadrant that lies on the hyperbola \( y^2 - x^2 = 6 \) and is closest to the point \( A(2,0) \). If we write the point as \( P(a,b) \), then

\[
a = \frac{1}{\sqrt{6}} \quad \text{and} \quad b = \frac{1}{\sqrt{6}}
\]

Want to minimize distance.

Now

\[
D = \sqrt{b^2 + (2-a)^2}
\]

But \((a,b)\) on \( y^2 - x^2 = 6 \) \( \Rightarrow b^2 - a^2 = 6 \)

\[
\Rightarrow b^2 = 6 + a^2
\]

\[
D = \sqrt{6 + a^2 + (2-a)^2}
\]

Next, \( s(t) = \sqrt{t} \text{ is } \uparrow \text{ for all } t > 0 \)

so \( \sqrt{f(t)} \text{ has max at } 6 \quad \Rightarrow \quad f(t) \text{ has min at } 0 \)

So, only need to minimize \( 6 + a^2 + (2-a)^2 \)

Now \( \frac{d}{da} \left( 6 + a^2 + (2-a)^2 \right) = 2a + 2(2-a)(-1) = 0 \Rightarrow a(2-a) = 2a\)

\( \Rightarrow 2 - a = a \Rightarrow 2 = 2a \Rightarrow a = 1 \).

Finally \( b^2 = 6 + a^2 \Rightarrow b^2 = 6 + 1 = 7 \Rightarrow b = \sqrt{7} \)
RELATED RATE PROBLEMS

► **Overall philosophy and recommended notation:** In a related rate problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). It is almost always better to use Leibniz’s notation \( \frac{dy}{dt} \), if we are differentiating, for instance, the function \( y \) with respect to time \( t \). The \( y' \) notation is more ambiguous when working with rates and should therefore be avoided.

► **Implicit derivatives:** Imagine you drop a rock in a still pond. This will cause expanding circular ripples in the pond. The area of the outer circle depends on the radius \( r \) of the perturbed area:

\[
A = \pi r^2.
\]

The radius of the outer circle depends on the amount of time \( t \) that has elapsed since you dropped the rock. Thus, the area also depends on time. In conclusion, it makes sense to find the rate of change of the area with respect to time and relate it to the rate of change of the radius with to time. We call it an *implicit derivative* as the function \( A \) is not explicitly given in terms of \( t \)...but only implicitly. We need the chain rule to do this.

► **Quick review of the chain rule:** Typically, we are given \( y \) as a function of \( u \) and \( u \) as a function of \( x \), so that we can think of \( y \) as a function of \( x \) also. The chain rule then says that

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.
\]

**Example 9:** Consider the area of a circle \( A = \pi r^2 \) and assume that \( r \) depends on \( t \). Find a formula for \( \frac{dA}{dt} \).

\[
\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}.
\]

Now \( A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r \)

\[
\Rightarrow \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}.
\]

► **Related rate guideline:** This guideline is found on pp. 143-144 of our textbook.

1. Read the problem quickly.
2. Read the problem carefully.
3. Identify the variables. Note that time is often an understood variable. If the problem involves geometry, draw a picture and label it. Label anything that does not change with a constant. Label anything that does change with a variable.
4. Write down which derivatives you are given. Use the units to help you determine which derivatives are given. The word “per” often indicates that you have a derivative.
5. Write down the derivative you are asked to find. “How fast...” or “How slowly...” indicates that the derivative is with respect to time.
6. Look at the quantities whose derivatives are given and the quantity whose derivative you are asked to find. Find a relationship between all of these quantities.
7. Use the chain rule to differentiate the relationship.
8. Substitute any particular information the problem gives you about values of quantities at a particular instant and solve the problem. To find all of the values to substitute, you may have to use the relationship you found in step 6. Take a snapshot of the picture at the particular instant.
**Example 10:** Boyle’s Law states that when a sample gas is compressed at a constant temperature, the pressure \( P \) and volume \( V \) satisfy the equation \( PV = c \), where \( c \) is a constant. Suppose that at a certain instant the volume is 600 cm\(^3\), the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

\[
\frac{d}{dt}(PV) = \frac{d}{dt}(\text{constant})
\]

\[
\frac{dP}{dt}V + P \frac{dV}{dt} = 0
\]

\[
ow, \quad V = 600, \quad P = 150
\]

\[
\frac{dP}{dt} = 20 \quad \Rightarrow \quad 20 \cdot 600 + 150 \frac{dV}{dt} = 0
\]

\[
\Rightarrow \quad 150 \frac{dV}{dt} = -12,000 \quad \Rightarrow \quad \frac{dV}{dt} = \frac{-12,000}{150} = -80
\]

Now, \( \frac{dV}{dt} = -80 \) so volume is increasing at rate of 80.

So Rate of decrease is 80.

**Example 11:** A train is traveling over a bridge at 30 miles per hour. A man on the train is walking toward the rear of the train at 2 miles per hour. How fast is the man traveling across the bridge in miles per hour?

Motion is occurring in one direction, so velocities simply add or subtract. Relative to a point on bridge, man is moving at 30 - 2 mph = 28 mph.

**Example 12:** Two trains leave a station at the same time. One travels north on a track at 30 mph. The second travels east on a track at 46 miles per hour. How fast are they traveling away from one another in miles per hour when the northbound train is 60 miles from the station?

\[
D = \sqrt{x^2 + y^2}
\]

\[
x = \text{distance eastbound train}
\]

\[
y = \text{distance northbound train}
\]

\[
\text{Given, } \frac{dx}{dt} = 46, \quad \frac{dy}{dt} = 30
\]

\[
\text{Want } \frac{dD}{dt}
\]

\[
\text{But } D^2 = x^2 + y^2, \quad \text{Take } \frac{d}{dt} \text{ on both sides}
\]

\[
2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}
\]

\[
\Rightarrow \quad \frac{dD}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{D}
\]

\[
\text{Now, } y = 60, \text{ travel at } 70 \text{ mph so two hours elapsed.}
\]

\[
\text{So } x = 46 \cdot 2 = 92
\]

\[
\text{So } \frac{dD}{dt} = \frac{92 \cdot 46 + 60 \cdot 30}{\sqrt{92^2 + 60^2}} \approx 54.92 \text{ mph.}
\]
Example 13: Two trains leave a station at 12:00 noon. One travels north on a track at 30 mph. The second travels east on a track at 80 miles per hour. At 1:00 PM the northbound train stops for one-half hour at a station while the eastbound train continues at 80 miles per hour without stopping. At 1:30 PM the northbound train continues north at 30 mph. How fast are the trains traveling away from one another at 2:00 PM?

\[ \frac{dy}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{\sqrt{x^2 + y^2}} \]

Sample 12 is still valid. Now, \( x \) travels for 2 hours, from noon to 2, so \( x = 2 \times 30 = 160 \) miles.

\[ \frac{dD}{dt} = \frac{160 \cdot 80 + 45 \cdot 30}{\sqrt{160^2 + 45^2}} \approx 87.13 \text{ mph} \]

Example 14: A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 feet/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?

\[ \frac{dx}{dt} = 1, \text{ want } \frac{dy}{dt} \]

Take \( \frac{d}{dt} \) on both sides of \( x^2 + y^2 = 10^2 \)

\[ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \]

\[ \frac{dy}{dt} = -\frac{2x}{y} \frac{dx}{dt} \]

\[ y \frac{dy}{dt} = -x \frac{dx}{dt} \]

\[ \Rightarrow \frac{dy}{dt} = -(x/y) \frac{dx}{dt} \]

\[ x = 6, y = \sqrt{100 - 6^2} = 8 \]

\[ \frac{dy}{dt} = -\left(\frac{6}{8}\right) \cdot 1 = -\frac{3}{4} \text{ ft/sec} \]

Example 15: A cylindrical water tank with its circular base parallel to the ground is being filled at the rate of 4 cubic feet per minute. The radius of the tank is 2 feet. How fast is the level of the water in the tank rising when the tank is half full? Give your answer in feet per minute.

\[ V = \pi r^2 h \]

\[ r = \text{constant} = 2 \text{ feet} \]

\[ V = \pi 2^2 h = 4\pi h \]

\[ 40 \frac{dV}{dt} = 4\pi \frac{dh}{dt} \]

But \( \frac{dV}{dt} = 4 \) (given)

\[ \frac{dh}{dt} = \frac{1}{\pi} \text{ ft/sec} \]
Example 16: A conical salt spreader is spreading salt at a rate of 3 cubic feet per minute. The diameter of the base of the cone is 4 feet and the height of the cone is 5 feet. How fast is the height of the salt in the spreader decreasing when the height of the salt in the spreader (measured from the vertex of the cone upward) is 3 feet? Give your answer in feet per minute. (It will be a positive number since we use the word “decreasing”.)

\[
V = \frac{1}{3} \pi r^2 h = 4 \Rightarrow \text{radius} = 2 \Rightarrow \text{base} = 2 \Rightarrow \text{cone is thinner near vertex}
\]

\[
\text{we need a relationship between } r + h.
\]

[Diagram of a cone with dimensions labeled]

\[
\frac{\text{Similar triangles}}{h} \Rightarrow \frac{r}{h} = \frac{2}{5} \Rightarrow r = \left(\frac{2}{5}\right)h
\]

\[
\text{so } V = \frac{1}{3} \pi \left(\frac{2}{5}h\right)^2 h = \frac{1}{3} \pi \cdot \frac{4}{25} \cdot h^3
\]

\[
V = \left(\frac{4 \pi}{175}\right) h^3
\]

\[
\text{so } \frac{dV}{dt} = 3 \left(\frac{4 \pi}{175}\right) h^2 \frac{dh}{dt} \Rightarrow \frac{dV}{dt} = \frac{12 \pi}{25} \cdot \frac{dh}{dt} = -1
\]

\[
\Rightarrow \frac{dh}{dt} = -\frac{25}{12 \pi} \Rightarrow \text{rate decrease of } h
\]

Example 17: It is estimated that the annual advertising revenue received by a certain newspaper will be

\[
R(x) = 0.5x^2 + 3x + 160
\]

thousand dollars when its circulation is \(x\) thousand. The circulation of the paper is currently 10,000 and is increasing at a rate of 2,000 papers per year. At what rate will the annual advertising revenue be increasing with respect to time 2 years from now?

\[
\text{Want } \frac{dR}{dt} \left|_{t=2}\right.
\]

\[
\text{Now } \frac{dR}{dt} = \frac{dR}{dx} \cdot \frac{dx}{dt}, \text{ and}
\]

\[
\frac{dR}{dx} = (5)x + 3 = x + 3
\]

\[
\text{so } \frac{dR}{dt} = (x + 3) \cdot \frac{dx}{dt},
\]

\[
x \text{ measured in 1000}, \text{ so } \frac{dx}{dt} = 2 \text{ of 2000 papers/year}
\]

\[
\Rightarrow \frac{dx}{dt} = 2
\]

\[
\Rightarrow x \big|_{\text{now}} = 10.
\]

\[
\text{But } x \rightarrow \text{ by 2 each year, so}
\]

\[
x \big|_{t=2} = 10 + 2 \cdot 2 = 14.
\]

\[
10 \frac{dR}{dt} = (14 + 3) \cdot 2 = 34
\]

\[
\text{Revenue increases at } \$ 34,000/\text{year}.
\]
Example 18: A stock is increasing in value at a rate of 10 dollars per share per year. An investor is buying shares of the stock at a rate of 26 shares per year. How fast is the value of the investor’s stock growing when the stock price is 50 dollars per share and the investor owns 100 shares? (Hint: Write down an expression for the total value of the stock owned by the investor.

\[
\frac{dV}{dt} = \frac{dn}{dt} \cdot p + n \cdot \frac{dp}{dt}
\]

Now, given
\[
\frac{dp}{dt} = 10, \quad \frac{dn}{dt} = 26, \quad p = 50, \quad n = 100
\]

So
\[
\frac{dV}{dt} = 26 \cdot 50 + 100 \cdot 10 = 2300 \text{/year}
\]

Example 19: Suppose that the demand function \( q \) for a certain product is given by

\[
q = 4000 e^{-0.01p},
\]

where \( p \) denotes the price of the product. If the item is currently selling for $100 per unit, and the quantity supplied is decreasing at a rate of 80 units per week, find the rate at which the price of the product is changing.

\[
\text{Want } \frac{dp}{dt}, \text{ change of price}
\]

Now given
\[
\frac{dq}{dt} = -80
\]

Also
\[
\frac{dq}{dt} = \frac{dq}{dp} \cdot \frac{dp}{dt},
\]

And
\[
\frac{dq}{dp} = \frac{d}{dp} \left[ 4000 e^{-0.01p} \right]
\]

\[
= (4000)(-0.01) e^{-0.01p} = -40 e^{-0.01p}
\]

\[
\text{So } \frac{dp}{dt} = 2 \cdot e^{-0.01} \text{5.44/week}
\]