MA 322: Matrix Algebra and Its Applications
Diagonalizability Example

If possible, diagonalize the matrix \( A = \begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \).

1. Just by looking at it, is \( A \) diagonalizable? What is the characteristic polynomial of \( A \)?

   \( A \) is triangular, so the eigenvalues lie on the diagonal.

   Since 2 is repeated, we need to check the dimension of the eigenspace to determine if \( A \) is diagonalizable.

   \( \det(A - \lambda I) = (\lambda - 5)(\lambda - 3)(\lambda - 2)^2 \).

2. Find a basis for \( \text{Eig}_5 \).

   \[
   A - 5I = \begin{bmatrix} 0 & -3 & 0 & 9 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
   \]

   \( \text{Eig}_5 = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \)

3. Find a basis for \( \text{Eig}_3 \).

   \[
   A - 3I = \begin{bmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
   \]

   \( \text{Eig}_3 = \text{Span} \left\{ \begin{bmatrix} 3/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \)

4. Find a basis for \( \text{Eig}_2 \).

   \[
   A - 2I = \begin{bmatrix} 3 & -3 & 0 & 9 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
   \]

   \( \text{Eig}_2 = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\} \)
5. Can we tell if $A$ is diagonalizable now? Find the diagonal matrix $D$ that is similar to $A$.

Yes, the sum of eigenspace dimensions is $n=4$.

$$D = \begin{bmatrix}
5 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix}$$

6. Find the matrix $P$ that contains an eigenbasis for $\mathbb{R}^4$, and calculate $P^{-1}$.

$$P = \begin{bmatrix}
1 & 3 & -1 & -1 \\
0 & 2 & -1 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad \begin{bmatrix} \lambda & I_4 \end{bmatrix} \sim \begin{bmatrix}
1 & -\frac{3}{2} & -\frac{1}{2} & 4 \\
0 & \frac{1}{2} & \frac{1}{2} & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

So, $P^{-1} = \begin{bmatrix}
1 & -\frac{3}{2} & -\frac{1}{2} & 4 \\
0 & \frac{1}{2} & \frac{1}{2} & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$

7. Use your answers to parts (5) and (6) to calculate $A^3$.

$$A^3 = (PD^3P^{-1})^3 = PD^3P^{-1}$$

$$D^3 = \begin{bmatrix}
5^3 & 0 & 0 & 0 \\
0 & 4^3 & 0 & 0 \\
0 & 0 & 3^3 & 0 \\
0 & 0 & 0 & 2^3
\end{bmatrix} = \begin{bmatrix}
125 & 0 & 0 & 0 \\
0 & 27 & 0 & 0 \\
0 & 0 & 8 & 0 \\
0 & 0 & 0 & 8
\end{bmatrix}$$

$$A^3 = \begin{bmatrix}
125 & -147 & -30 & 411 \\
0 & 27 & 19 & -38 \\
0 & 0 & 8 & 0 \\
0 & 0 & 0 & 8
\end{bmatrix}$$

$$\begin{bmatrix}
1 & 3 & -1 & -1 \\
0 & 2 & -1 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
125 & 0 & 0 & 0 \\
0 & 27 & 0 & 0 \\
0 & 0 & 8 & 0 \\
0 & 0 & 0 & 8
\end{bmatrix} \begin{bmatrix}
1 & -\frac{3}{2} & -\frac{1}{2} & 4 \\
0 & \frac{1}{2} & \frac{1}{2} & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = A^3$$