1. **Is the vector** \( \mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \) **in** \( \text{Span} \left\{ \begin{bmatrix} -4 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} \)?

**Solution:** Consider the augmented matrix

\[
A = \begin{bmatrix} 2 & -4 & 1 \\ 3 & 7 & 2 \\ 1 & 8 & 3 \end{bmatrix}
\]

\( A \) is row equivalent to the identity matrix

\[
I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Therefore the three vectors are linearly independent and so \( \mathbf{w} \) is not in

\[
\text{Span} \left\{ \begin{bmatrix} -4 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}.
\]

2. **Determine if the columns of** \( \mathbf{A} \) **form a linearly independent set.**

\[
A = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}
\]

**Solution:** \( A \) is row equivalent to

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},
\]

thus its columns form a linearly independent set.

3. **For what values of** \( h \) **is** \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) **linearly independent?**

\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ -14 \\ h \end{bmatrix}
\]

**Solution:** Consider the matrix \( A \) with columns \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \). \( A \) is row equivalent to the identity matrix, regardless of \( h \). Therefore \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) is a linearly independent set for every value of \( h \).
4. Prove that if \( S = \{v_1, v_2, \ldots, v_p\} \) is a linearly dependent set of vectors in \( \mathbb{R}^n \), then there exists \( v_k \) in \( S \) such that \( \text{Span}(S \setminus \{v_k\}) = \text{Span}(S) \).

Solution: \( S \) is linearly dependent. Therefore there exists some solution to
\[
x_1v_1 + \ldots + x_pv_p = 0
\]
and some \( k \) such that \( x_k \neq 0 \). Then
\[
-x_kv_k = x_1v_1 + \ldots + x_{k-1}v_{k-1} + x_{k+1}v_{k+1} + \ldots + x_pv_p
\]
and hence
\[
v_k = \frac{x_1}{x_k}v_1 - \ldots - \frac{x_{k-1}}{x_k}v_{k-1} - \frac{x_{k+1}}{x_k}v_{k+1} - \ldots - \frac{x_p}{x_k}v_p
\]
For any \( w \in S \), we have
\[
w = w_1v_1 + \ldots + w_pv_p
\]
\[
= w_1v_1 + \ldots + w_k \left( \frac{x_1}{x_k}v_1 - \ldots - \frac{x_{k-1}}{x_k}v_{k-1} - \frac{x_{k+1}}{x_k}v_{k+1} - \ldots - \frac{x_p}{x_k}v_p \right) + \ldots + w_pv_p
\]
Hence, it is also true that \( w \in \text{Span}(S \setminus \{v_k\}) \). Thus, \( \text{Span}(S \setminus \{v_k\}) = \text{Span}(S) \).

5. Suppose \( S = \{v_1, v_2, \ldots, v_p\} \) is a set of vectors in \( \mathbb{R}^n \). If \( p > n \), prove that \( S \) is a linearly dependent set using only what you know about span and row reduction.

Proof: Given the \( n \times p \) matrix \( A \) whose columns are the vectors \( \{v_1, v_2, \ldots, v_p\} \), the columns of \( A \) are linearly independent if and only if \( A \) is row equivalent to a matrix with \( p \) pivot columns. \( A \) can have at most \( \min(n, p) \) pivot columns. Therefore, if \( p > n \), the number of pivot columns is less than \( p \), which is the number of vectors. Thus, \( \{v_1, v_2, \ldots, v_n\} \) is a linearly dependent set.

6. Suppose \( A \) is an \( m \times n \) matrix with the property that for all \( b \) in \( \mathbb{R}^m \) the equation \( Ax = b \) has at most one solution. Use the definition of linear independence to explain why the columns of \( A \) must be linearly independent.

Solution: Given \( Ax = b \) has at most one solution, in particular \( Ax = 0 \) has at most one solution (and hence exactly one since \( x = 0 \) is clearly a solution). If \( \{v_1, v_2, \ldots, v_n\} \) are the columns of \( A \), then \( x_1v_1 + \ldots + x_nv_n = 0 \) has exactly one solution. Thus, the columns of \( A \), \( \{v_1, v_2, \ldots, v_n\} \), are linearly independent.