AN INTRODUCTION TO RESONANCES AND THE SCATTERING PHASE: NOTES FOR TALKS AT THE UNIVERSITY OF KENTUCKY, JUNE 2002

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1. Introduction

The spectrum of the Laplacian on a smooth compact manifold is a very interesting object. However, if we consider, for example, the Laplacian $\Delta \geq 0$ with Dirichlet boundary conditions on $\mathbb{R}^n \setminus \{\mathcal{O}\}$, with $\mathcal{O}$ compact, the spectrum is $[0, \infty)$: not very interesting. We discuss two "replacements" for the spectrum: resonances and the scattering phase.

These notes owe a great deal to the surveys [22] and [19]. The note [23] provides a light-hearted introduction to resonances. For a non-technical survey of scattering theory in general, from a rather geometric point of view, see [7]. A more recent survey is [25]. We recommend these for further reading, and further references, for the interested reader.

2. Disclaimers

These notes are meant to be a quick introduction to resonances and the scattering phase for those who are more familiar with spectral theory on compact manifolds. We emphasize the scalar Euclidean setting, though we make some incursions into hyperbolic scattering and some other settings.

Many things are necessarily omitted from these notes. We do not attempt to provide a complete history or any proofs of results, and we do not discuss any of the large body of semiclassical results. Moreover, time constraints force us to omit many other interesting results. The bibliography is extremely incomplete (and has been added to in an unsystematic fashion), though if time allows I may improve it. There are certainly typos and other errors; I would appreciate hearing about them so that I may correct them.

3. The black box setting

The operators we consider are of the general type considered by Sjöstrand and Zworski in [13]. We give the assumptions here, using much of the notation of their paper.

For $R \in \mathbb{R}_+$, set $B(R) = \{x \in \mathbb{R}^n : |x| < R\}$. Let $\mathcal{H}$ be a complex Hilbert space with orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_1} \oplus L^2(\mathbb{R}^n \setminus B(R_1)).$$

Following [13], we denote the corresponding orthogonal projections by $u \mapsto u_{|B(R_1)}$ and $u \mapsto u_{|\mathbb{R}^n \setminus B(R_1)}$. We assume that the operator $P : \mathcal{H} \to \mathcal{H}$ is lower semi-bounded [1], self-adjoint with domain $\mathcal{D} \subset \mathcal{H}$. Furthermore, if $u \in H^2(\mathbb{R}^n \setminus B(R_1))$ and u vanishes near $\overline{B(R_1)}$, then $u \in \mathcal{D}$; and conversely $\mathcal{D}_{|\mathbb{R}^n \setminus B(R_1)} \subset H^2(\mathbb{R}^n \setminus B(R_1))$. The operator P is the (positive) Laplacian Δ outside $B(R_1)$:

$$Pu_{|\mathbb{R}^n \setminus B(R_1)} = \Delta u_{|\mathbb{R}^n \setminus B(R_1)} \text{ for all } u \in \mathcal{D}$$

and

(1) $$\mathbf{1}_{B(R_1)}(P+i)^{-m_0} \text{ is trace class for some finite } m_0.$$ [2]

Examples. If this all seems a bit abstract, it can be helpful to keep in mind the following examples:

- $P = \Delta + V$, with $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R})$ (potential scattering).
- P is the Laplacian, with Dirichlet or Neumann boundary conditions, on $\mathbb{R}^n \setminus \mathcal{O}$, with $\mathcal{O}$ compact (and with appropriate regularity of $\mathcal{O}$). (Obstacle scattering.)
- P is the Laplacian of some compactly supported metric perturbation of $\mathbb{R}^n$.
- P is the Laplacian outside a compact set, but is a nice hypoelliptic differential operator. [3]
- P is a combination of the above.
- P is $\Delta - 1/4$ on a noncompact, finite volume surface X with one or more hyperbolic cusps. In this case $n = 1$ (and we use a half-line rather than $\mathbb{R} \setminus B(R_1)$) and $\mathcal{H}_{R_1}$ consists of square-integrable functions on X whose 0th Fourier coefficient vanishes for $y > a > 0$.

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[1] This condition can be omitted with some care.

[2] This condition is overly restrictive for some results. For some, it suffices to have $\mathbf{1}_{B(R_1)}(P+i)^{-1}$ compact.

[3] Here is an example, taken from [14]. Let (ρ, θ) be polar coordinates on $\mathbb{R}^n$, and let Δ_θ be the Laplacian on S^{n-1}. Let

$$P = -\frac{\partial^2}{\partial \rho^2} - \frac{n-1}{\rho}\frac{\partial}{\partial \rho} + f(\rho)\frac{(\rho-1)^2}{\rho^2}\Delta_\theta$$

where $f \in C^\infty(\mathbb{R})$, $f(\rho) = (\rho-1)^{-2}$ when $\rho \notin [1/4, 2]$ and $f(\rho) = 1$ for $\rho \in [1/2, 3/2]$. Then the associated eigenvalue counting function (defined a little later) $\Phi(r) = Ar^{2(n-1)} + o(r^{2n-2})$. A more general example can be found in [14].

4. Why Resonances?

One way to motivate the study of resonances is by analogy to eigenvalues. Let $R(\lambda) = (P - \lambda^2)^{-1}$ when $\Im\lambda > 0$, and let $\chi \in C_c^\infty(\mathbb{R}^n)$ be 1 for $|x| \leq R_1$. Our assumptions on P above guarantee that $\chi R(\lambda)\chi$ [4] has a meromorphic continuation to $\mathbb{C}$ if n is odd, and to the logarithmic surface Λ if n is even. The poles of $\chi R(\lambda)\chi$ in the lower half plane are called *resonances* [5]. The poles are independent of the choice of cut-off function χ having these properties. Thus we see the first similarity between eigenvalues and resonances: both are poles of the resolvent. We will denote the poles of $\chi R(\lambda)\chi$ (for our operator P) by $\mathcal{R}$. We explicitly include the poles of $\chi R(\lambda)\chi$ corresponding to eigenvalues. Resonances and eigenvalues are always repeated according to their multiplicities. However, we will occasionally need $\operatorname{mult}\lambda_j$, which for $\lambda_j \in \mathcal{R}$ we define to be

$$\operatorname{mult}\lambda_j = \#\{\lambda_k \in \mathcal{R} : \lambda_k = \lambda_j\}.$$

Another way to see the similarities between eigenvalues and resonances is via a trace formula. For comparison, let M be a smooth, compact connected Riemannian manifold and let Δ_M be the Laplacian on M. Then, as a distribution,

$$\operatorname{tr}\cos(t\sqrt{\Delta_M}) = \frac{1}{2}\sum_{\sigma_j^2 \in \operatorname{spec}\Delta_M} e^{-i\sigma_j t} + \frac{1}{2}.$$

Compare, if n is odd, $0 \notin \mathcal{R}$,

$$\operatorname{tr}_r(\cos(t\sqrt{P}) - \cos(t\sqrt{\Delta_{\mathbb{R}^n}})) = u(t) = \frac{1}{2}\sum_{\lambda_j \in \mathcal{R}} e^{-i\lambda_j|t|},\ t \neq 0 \tag{2}$$ [6]

(Lax-Phillips, Bardos-Guillot-Ralston, Melrose, Sjöstrand-Zworski, SáBarreto-Zworski). This is often called the Poisson formula. [7]

Here is another similarity: If $(D_t^2 - \Delta_M)u_M(t) = 0$, $u_M(0)$, $\frac{d}{dt}u_M(0) \in C^\infty(M)$, then we can expand u_M in terms of the eigenvalues of Δ_M:

$$u_M(t,x) = \sum_{\sigma_j^2 \in \operatorname{spec}(\Delta_M)} e^{-i\sigma_j t} g_{\sigma_j}(x).$$

A similar expansion exists for "non-trapping" scattering settings, if n is odd: If $(D_t^2 - P)u(t) = 0$, $u(0)$, $\frac{d}{dt}u(0)$ are smooth and compactly supported, then, for χ

[4] If $u_1 \in \mathcal{H}_{R_1}$, and $\chi = 1$ on $\overline{B(R_1)}$, we understand $\chi u_1 = u_1$.

[5] Some may include square roots of eigenvalues as resonances as well.

[6] In case $\mathcal{H} = L^2(\mathbb{R}^n)$, $tr_r = tr$. In general, fix $\chi \in C_c^\infty(\mathbb{R}^n)$ so that $\chi \equiv 1$ on $\overline{B(R_1)}$ Then $\operatorname{tr}_r(f(\sqrt{P}) - f(\sqrt{\Delta_{\mathbb{R}^n}})) = \operatorname{tr}(\chi f(\sqrt{P})) - \operatorname{tr}(\chi f(\sqrt{\Delta_{\mathbb{R}^n}})) + \operatorname{tr}((1-\chi)(f(\sqrt{P}) - f(\sqrt{\Delta_{\mathbb{R}^n}}))$.

[7] This formula has proved occasionally helpful in studying the distribution of resonances. For example, in many settings an expansion of the left hand side near $t = 0$ is known, and in some cases has proved useful in giving lower bounds on the number of resonances. In other cases, knowledge of singularities of the left hand side away from $t = 0$ have led to other lower bounds.

compactly supported,

$$\chi u(t) = \sum_{\substack{\lambda_j \in \mathcal{R} \\ \Im \lambda_j > -\beta}} e^{-i\lambda_j t} \sum_{k=0}^{\mathrm{mult}(\lambda_j)-1} t^k f_{j,k}(x) + \sum_{\substack{\sigma_j^2 \text{ eigenvalues of } P \\ \Im \sigma_j < 0}} e^{-i\sigma_j t} f_j(x) + O(e^{-\beta t})^8$$

(Lax-Phillips, Vainberg). (In section 7 we define "non-trapping".)

Equation (4) helps with a dynamical understanding of resonances: if λ_j is a resonance, we can think of $\Re\lambda_j$ as corresponding to the energy of a wave, and $|\Im\lambda_j|$ to its rate of decay. The larger $|\Im\lambda_j|$, the faster the wave decays. Thus resonances near the real axis are in general considered more physically relevant than those far from the real axis.

Additional motivation for the study of resonances comes from the fact that embedded eigenvalues[9] tend to be unstable– that is, they can "dissolve fairly easily into resonances" that do not correspond to eigenvalues– see [4, 10], for example. Another reason to be interested in resonances is that for the modular surface the scattering poles (poles that do not correspond to eigenvalues) correspond to the non-trivial zeros of the Riemann zeta function.

5. Global upper bounds

For odd dimensional Euclidean scattering, one analog for the eigenvalue counting function is the counting function for the number of resonances inside a ball of radius r:

$$N_\rho(r) = \#\{\lambda_j \in \mathcal{R} : |\lambda_j| \leq r\}.$$

For even dimensions things are a bit trickier: one choice is

$$N_\rho(r,a) = \#\{\lambda_j \in \mathcal{R} : |\lambda_j| \leq r,\ -a \leq \arg\lambda_j \leq 0 \text{ or } \pi \leq \arg\lambda_j \leq \pi + a\}.$$

To describe the upper bounds on $N_\rho(r)$ or $N_\rho(r,a)$, we introduce a reference operator. Let $R_2 > R_1$ be fixed. Let $P^\#$ be the operator with Dirichlet boundary conditions at $|x| = R_2$ on

$$\mathcal{H} \oplus L^2(B(R_2) \setminus B(R_1)).$$

Our assumptions on P ensure that $P^\#$ has discrete spectrum (bounded below). We assume

$$\Phi(r) = \#\{\nu_l :\ \nu_l \in \mathrm{spec}(P^\#),\ |\nu_l| \leq r^2\}$$

satisfies, for some $C > 0$, $\delta > 0$,

$$\Phi(\theta r) \leq C\theta^\delta \Phi(r),\ r\theta >> 1, 0 < \theta < 1,\ \Phi(r) \geq C^{-1} r^n.^{10}$$

[8] Such an expansion is useful in computing resonances, or in target identification.

[9] That is, embedded in the continuous spectrum.

[10] Examples of such Φ are $\Phi(r) = c_1 r^{n_\#} + o(r^{n_\#})$, or $\Phi(r) = c_1 r^{n_\#}(\log r)^p + o(r^{n_\#}(\log r)^p)$, both with $n_\# \geq n$.

Under these assumptions, [11]

$$i) \text{ for } n \text{ odd, } N_\rho(r) \leq C\Phi(Cr),\ r \geq 1$$
$$ii) \text{ for } n \text{ even, } N_\rho(r,a) \leq C\langle a\rangle[(\log\langle a\rangle)^n + \Phi(Cr)],\ r \geq 1.$$

The first bound is due to Sjöstrand-Zworski [13] in this generality (with another proof later due to Vodev [15]), and the second to Vodev [16]. [12] The Sjöstrand-Zworski proof uses complex scaling, and the Vodev proof the Fredholm determinant method (initiated by Melrose and also developed by Zworski). The idea is roughly this, though there are refinements: the poles of $\chi R(\lambda)\chi$ are contained in the zeros of $I + K(\lambda)$, where $K(\lambda)$ is a compact, holomorphic operator. Moreover, for some m, $K(\lambda)^m$ is trace class, so that the poles of $\chi R(\lambda)\chi$ are contained in the zeros of $\det(I + (-1)^{m-1}(K(\lambda))^m)$. With a bound on $|\det(I + (-1)^{m-1}(K(\lambda))^m)|$ (this takes work!) one can apply Jensen's or Carleman's theorem to obtain an upper bound on the number of zeros of $\det(I + (-1)^{m-1}(K(\lambda))^m)$.

While not a "global" upper bound, recent results of Petkov-Zworski and Burq give bounds near the real axis. Suppose $\Phi(r) \sim cr^n + O(r^{n-1})$. Then, for $\lambda \in \mathbb{R}$,

$$\#\{\lambda_j \in \mathcal{R} : |\lambda_j - \lambda| < \alpha\} = O_\alpha(\lambda^{n-1}) \text{ as } \lambda \to \infty.$$

(That is, near the real axis at least, there is the same sort of upper bound as comes from Weyl's law for eigenvalues.)

6. Lower bounds and asymptotics

In only a few situations are asymptotics of the resonance counting function known.

- One-dimensional potential scattering: $P = -\frac{d^2}{dx^2} + V(x)$, $V \in L^\infty_{\text{comp}}(\mathbb{R};\mathbb{R})$. Let the convex hull of the support of V be $[a,b]$. Then

  $$N_\rho(r) = \frac{2}{\pi}|b-a|r + o(r)$$

  (Regge, Zworski, Froese). [13] It is also known that "most" of the resonances lie in arbitrarily small sectors about the real axis; that is, in a sector of radius r that does not meet the real axis, there are $o(r)$ resonances.
- Degeneracies: If $\Phi(r) = \varphi(r)(1+o(1))$, $\lim_{r\to\infty} \frac{r^n}{\varphi(r)} = 0$, then

  $$\#\{\lambda_j \in \mathcal{R} : 0 > \arg\lambda_j > -\theta,\ |\lambda_j| < r\} = \varphi(r)(1+o(1))$$

[11] These results illustrate Peter's point that what happens on a compact set is everything.

[12] Earlier results in specific situations are due to Melrose, Zworski, Intissar, Sjöstrand, and Vodev, among others.

[13] In fact, Froese has some results for superexponentially decaying potentials.

[11]. Since in odd dimensions, $n \geq 3$ there is the bound

$$\sum_{\lambda_j \in \mathcal{R}:\, |\lambda_j| \leq r} \frac{|\Im \lambda_j|}{|\lambda_j|^2} \leq C r^{n-1},$$

[17, 9], this gives the asymptotics $N_\rho(r) = 2\varphi(r)(1+o(1))$ (Earlier results of Vodev). Again, "most" resonances lie in sectors about the real axis.

- The transmission problem: Let $\mathcal{O}$ be a compact strictly convex domain with smooth boundary. Let $c \neq 1$ and α be two positive constants. Then, if $(u_1, u_2) \in \mathcal{D} \subset L^2(\mathcal{O}; \alpha^{-1} c^{-2} dx) \oplus L^2(\mathbb{R}^n \setminus \mathcal{O})$, then $Pu = (c^2 \Delta u_1, \Delta u_2)$, with

$$\mathcal{D} = \{(u_1, u_2) : u_1 \in H^2(\mathcal{O}),\ u_2 \in H^2(\mathbb{R}^n \setminus \mathcal{O}), \\ (u_1)_{|\partial\mathcal{O}} = (u_2)_{|\partial\mathcal{O}},\ (\partial_{\nu'} u_1)_{|\partial\mathcal{O}} = -\alpha(\partial_\nu u_2)_{|\partial\mathcal{O}}\}$$

where ν is the outward unit normal to $\partial\mathcal{O}$ and $\nu' = -\nu$ is the inner one. Then if $c < 1$ there is a constant $\alpha_0 > 0$ so that if $\alpha \leq \alpha_0$,

$$\#\{\lambda_j \in \mathcal{R} :\ 0 > \Im \lambda_j \geq -C\alpha,\ |\lambda_j| \leq r\} = \tau_n c^{-n} \operatorname{vol}(\mathcal{O}) r^n + O_\epsilon(r^{n-1/3+\epsilon}),\ \forall \epsilon > 0$$

for some positive constant C independent of α (Cardoso-Popov-Vodev). [14] There is a related result (with similar asymptotics) if $c > 1$.

- For comparison, for X a finite-volume Riemannian surface with cusp ends, there are the asymptotics:

$$N_\rho(r) = \frac{\operatorname{vol}(X)}{2\pi} r^2 + o(r^{3/2+\epsilon})$$

(Selberg, Müller, Parnovski) where we remind the reader that this count includes the (positive and negative) square roots of embedded eigenvalues.

In certain specific examples, there are lower bounds that prove the optimality of the order of the corresponding upper bound (For example, in potential scattering).

There are other lower bounds:

- For obstacle scattering (with Dirichlet or Neumann boundary conditions), or for potential scattering, where V has fixed sign (all in odd dimensions): there is a $c,\ c_0 > 0$ so that

$$\#\{\lambda_j \in \mathcal{R} : \lambda_j \in i\mathbb{R},\ |\lambda_j| \leq r\} \geq c r^{n-1} - c_0$$

(Lax-Phillips, Vasy).

- For $V \in C_c^\infty(\mathbb{R}^n)$, $V \not\equiv 0$, n odd

$$\limsup_{r\to\infty} \frac{N_\rho(r)}{r} > 0$$

[14] In fact, they obtain a pole-free region as well, and we have not included the full generality of their result.

(This is due to Sá Barreto; earlier related results are due to Melrose, Sá Barreto-Zworski, Christiansen.)

- For $V \in C_c^\infty(\mathbb{R}^n)$, $V \not\equiv 0$, n even
$$\limsup_{r\to\infty} \frac{\#\{\lambda_j \in \mathcal{R} : \frac{1}{r} \le |\lambda_j| \le r,\ |\arg \lambda_j| < \log r\}}{\log r (\log\log r)^{-p}} > 0$$
for any $p > 1$. (This is due to Sá Barreto; earlier results of Sá Barreto-Tang.)
- There are other settings in which it is known that there exist infinitely many resonances. Some of these are a class of superexponentially decaying perturbations of the Laplacian on $\mathbb{R}^3$ (Sá Barreto-Zworski), metric perturbations under certain conditions (in odd dimensions Sá Barreto-Tang, and in even dimensions Tang).
- Recall we defined the distribution $u(t)$ in (2). A result of Sjöstrand-Zworski uses singularities of $u(t)$ away from $t = 0$ to obtain lower bounds on the number of resonances: Suppose that there is a $d > 0$ and a function $\phi_d \in C_0^\infty(0,\infty)$, $\phi_d = 1$ in a neighborhood of d, such that
$$\widehat{\phi_d u}(\lambda) \ge b(1 - o(1))\lambda^k,\ \lambda \gg 1,\ b > 0.$$
Then there is a $\gamma > 0$ so that if $k \ge 0$
$$\#\{\lambda_j \in \mathcal{R} : \Im\lambda_j \ge -\gamma \log|\lambda_j|,\ |\lambda_j| \le r\} \ge \frac{b(1-o(1))}{2\pi(k+1)} r^{k+1},\ r \gg 1.$$
If $k < 0$, then for every $\delta > 0$, there is a $r_0(\delta) > 1$ so that
$$\#\{\lambda_j \in \mathcal{R} : \Im\lambda_j \ge -\gamma \log|\lambda_j|,\ |\lambda_j| \le r\} \ge r^{1-\delta},\ r \ge r_0(\delta).$$
- A geometric-type relation involves lower bounds coming from "many" periodic orbits in certain classes of metric scattering. Let $\Pi(T) \subset \Sigma = \{(x,\xi) \in T^*(\mathbb{R}^n \setminus \mathcal{O}) : |\xi|_x = 1\}$ be the set of all periodic trajectories with period $T \neq 0$, and let $d\mu$ be the Liouville measure on Σ. Then there is the following result of Popov:
Suppose that $\mu(\Pi(T_0)) > 0$ for some $T_0 > 0$. Then for every $\gamma > 0$, we have
$$N_\gamma(r) \ge \frac{\mu(\Pi(T_0))}{n(2\pi)^n}(1 - o(r))r^n,\ r \gg 1$$
where
$$N_\gamma(r) = \#\{\lambda_j \in \mathcal{R} :\ -\pi/2 \le \arg\lambda \le 0,\ -\Im\lambda \le \gamma \log|\lambda|\}.$$
(The proof uses some results of [12] and the Sjöstrand-Zworski results described in the previous item.) Petkov-Zworski obtained a lower bound of the same order (but with worse constant) for the resonances $\#\{\lambda_j \in \mathcal{R} :$

$|\Re\lambda_j| \leq r,\ \Im\lambda_j \geq -\epsilon\}$. Moreover, they obtained results about the clustering of resonances, similar to results on the clustering of eigenvalues. (These results might belong in the next section.)

- The existence of infinitely many quasimodes implies the existence of a sequence of resonances tending to the real axis. This was first proved by Stefanov-Vodev in odd dimensions, extended to even dimensions (and non-compact perturbations) by Tang-Zworski (with some lower bounds on the number of resonances), and then improved upon (bounding from below the number of resonances in terms of the number of quasimodes) by Stefanov. Results of Popov then give a sharp (in order) lower bound on the number of resonances near the real axis under the assumption of the existence of certain elliptic periodic broken rays.

7. Resonances and geometry

In this section we discuss several results connecting the distribution of resonances with the underlying geometry of a space. Let's suppose that P corresponds to potential, metric, or obstacle scattering (or a combination).

The "quantum" non-trapping condition for an operator P is that for any $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi \equiv 1$ in a neighbourhood of $B(R_1)$, there is a $T = T_{\chi,P}$ so that $\chi \cos(t\sqrt{P})\chi$ is a smoothing operator for any $t > T$. For obstacle scattering, the "classical non-trapping" condition is that any unit speed generalized geodesic, propagating according to laws of geometrical optics, which is inside $B(R_2)$, $R_2 > R_1$, at time 0 leaves after a fixed time T. That the classical condition implies the quantum condition is the work of many authors, including Andersson, Melrose, Morawetz, Ralston, Strauss, Sjöstrand and Taylor – see the appendix to the second edition of [6] and references given there.

If P is "non-trapping" and n is odd, then for any N, there are only finitely many resonances above the line $\Im\lambda = -N(1 + \log(1 + |\Re\lambda|))$ (Lax-Phillips, Vainberg, Melrose). [15] For odd dimensions, the Lax-Phillips conjecture is that for any trapping obstacle, there is a sequence λ_j of resonances with $\Im\lambda_j \uparrow 0$. This is untrue, as shown by the example of scattering by two strictly convex bodies, where the resonances approximate a lattice (Ikawa, Gerard). There are (at least) two "modified Lax-Phillips conjectures" (for Euclidean scattering, n odd):

- For any trapping obstacle, there is a strip $\Im\lambda > -C$ with infinitely many resonances.
- For any trapping obstacle, there is an N with infinitely many resonances above $\Im\lambda = -N(1 + \log(1 + |\Re\lambda|))$.

[15] For contrast, compare the results of Popov and Petkov-Zworski mentioned in the previous section on lower bounds near the real axis in the presence of "many" periodic orbits of the same period.

Another relation between geometry and the distribution of resonances involves obstacle scattering in odd dimensions. If $\mathcal{O}$ is a non-trapping obstacle with analytic boundary, then for the Laplacian with Dirichlet boundary conditions in the exterior of $\mathcal{O}$ there are constants C_1, C_2 (depending on the obstacle) such that there are no resonances in the region

$$\Im\lambda \geq -C_1|\lambda|^{1/3} + C_2$$

(Bardos-Lebeau-Rauch). For strictly convex obstacles with C^∞ boundary, a similar result was proved by Hargé-Lebeau, with possibly smaller constant C_1.

8. Why the scattering phase?

The second analog to the eigenvalue counting function that we discuss is the scattering phase. To motivate its study, let's return to M, our smooth, compact manifold with Laplacian Δ_M. Let

$$N_M(\lambda) = \#\{\lambda_j \in \sigma(\Delta_M) : \ |\lambda_j| \leq \lambda^2\}.$$

If $f \in \mathcal{S}(\mathbb{R})$ and f is supported in $\lambda > 0$,

$$\operatorname{tr} f(\Delta_M) = \int_0^\infty f(\lambda^2) \frac{d}{d\lambda} N_M(\lambda) d\lambda.$$

On the other hand, for the same class of f, we have the Birman-Krein formula

$$\operatorname{tr}_r(f(P) - f(\Delta_{\mathbb{R}^n})) = \int_0^\infty f(\lambda^2) \frac{d}{d\lambda}(\sigma(\lambda) + N_e(\lambda)) d\lambda.^{16}$$

Here

$$N_e(\lambda) = \#\{\sigma_j^2 \text{ eigenvalues of } P : \ \sigma_j^2 \leq \lambda^2\}$$

and $\sigma(\lambda) = \frac{1}{2\pi i} \log \det S(\lambda)$, where $S(\lambda)$ is the scattering matrix. Recall that $S(\lambda) = I + A(\lambda)$, where $A(\lambda)$ is trace class (actually smoothing), so the determinant is well-defined. Additionally, $S(\lambda)$ is unitary for $\lambda \in \mathbb{R}$, so that $\sigma(\lambda)$ is real for $\lambda \in \mathbb{R}$.[17]

9. Weyl asymptotics of the scattering phase

For the Laplacian on M, our smooth compact manifold without boundary, we have the well-known Weyl asymptotics:

$$N_M(\lambda) = c_n \operatorname{vol}(M) \lambda^n + O(\lambda^{n-1}), \ \lambda \to \infty,$$

where n is the dimension of M. If we try to do something analogous for our operator P, the first thing we notice is that we must consider the sum $\sigma(\lambda) + N_e(\lambda)$. (In many, but not all, scattering settings, $N_e(\lambda)$ is "small" and can be ignored.)

[16] You will see different constants in front of σ, depending on normalization and choice of physical half plane ($\Im\lambda > 0$ or $\Im\lambda < 0$).

[17] We remark here that the scattering phase determines $\{\lambda_j \in \mathcal{R} : \Im\lambda_j \neq 0\}$.

Finding Weyl asymptotics for P is more complicated than the problem for Δ_M, but nonetheless in many settings asymptotics are known. [18]

For potential scattering on $\mathbb{R}^n$, if the potential V is smooth,

$$\sigma(\lambda) \sim \sum_{j=0}^{\infty} \alpha_j \lambda^{n-2j-2} + c_0$$

where for n even $\alpha_j = 0$ for $j > (n-2)/2$. [19] (Buslaev, Majda-Ralston, Colin de Verdière, Guillopé, Popov).

For a non-trapping obstacle, Petkov and Popov [8] obtained an asymptotic expansion of the scattering phase:

$$\sigma(\lambda) \sim c_n \operatorname{vol}(\mathcal{O})\lambda^n \pm \alpha_2 \operatorname{vol}(\partial\mathcal{O})\lambda^{n-1} + O(\lambda^{n-2})^{20}$$

where one takes the + sign for the Dirichlet problem and "−" for the Neumann problem.

A general result that relates the asymptotics for the eigenvalue counting function of the reference operator $P^{\#}$ to the asymptotics for $\sigma(\lambda) + N_e(\lambda)$ is the following. Suppose that $n \geq 2$ and $\Phi(\lambda) - \Phi(\lambda - 1) = \mathcal{O}(\lambda^d)$ for some finite d. Then, as $\lambda \to \infty$,

$$N_e(\lambda) + \sigma(\lambda) = \Phi(\lambda) - c_n \operatorname{vol} B(R_2)\lambda^n + \mathcal{O}(\lambda^{\max(n-1,d)})$$

where c_n is the Weyl constant (Christiansen, based on techniques of Robert).

In addition to the results already mentioned, there are many other (many earlier) results on the asymptotics of the scattering phase, due to, among others, Buslaev, Jensen-Kato, Majda-Ralston, Melrose, Popov, Robert, Sjöstrand-Zworski.

Asymptotics of the scattering phase, or scattering phase plus eigenvalue counting function, are also known for the Laplacian on certain classes of manifolds. We include a brief listing here to give an idea of some of the types of manifolds on which scattering theory has been studied. [21] These include finite-volume manifolds with cusp ends (Selberg, Müller, Parnovski, Christiansen (asymptotically cusp)), manifolds with cylindrical ends (Christiansen-Zworski, Parnovski), surfaces with hyperbolic ends (not necessarily finite volume: Guillopé-Zworski), and asymptotically Euclidean manifolds (Christiansen, Parnovski). In general the leading term is $c_{\dim X} \operatorname{vol}_r(X)\lambda^{\dim X}$, where X is the manifold and $\operatorname{vol}_r(X)$ is its (regularized, if necessary) volume.

[18] The reason that this is harder than obtaining Weyl asymptotics for $P^\sharp$ is that $\sigma(\lambda) + N_e(\lambda)$ need not be monotone, unlike $\Phi(\lambda)$. Therefore, the Tauberian theorems often used in proving Weyl asymptotics cannot be directly applied.

[19] $\alpha_0 = \alpha'_{0,n} \int V(x)dx$, $\alpha_1 = \alpha'_{1,n} \int |V(x)|^2 dx$.

[20] In fact they obtained $\frac{d}{d\lambda}\sigma(\lambda) \sim \sum_{j=0}^{\infty} \tilde{\alpha}_j \lambda^{n-1-j}$.

[21] This could be considered an illustration of Peter's point that what happens at infinity is everything, since a good understanding of what happens at infinity can allow one to develop a scattering theory.

Unless there are only a finite number of eigenvalues, in general it is not possible to separately find the asymptotics of the eigenvalue counting function and the scattering phase. However, there are some results in this area for certain finite volume hyperbolic surfaces. [22]

10. Resonances and the scattering phase

In odd dimensions, there is a nice relationship, proved by Zworski, between the resonances and $s(\lambda) = \det S(\lambda)$. (Of course, $\sigma(\lambda) = (2\pi i)^{-1} \log s(\lambda)$.)

Recall that we defined a reference operator $P^{\#}$ in Section 5. Suppose n is odd and $\Phi(r)$, the eigenvalue counting function for $P^{\#}$, satisfies $\Phi(r) \leq C_\epsilon(1+r^{m+\epsilon}),\ n \leq m$ for some m and for any $\epsilon > 0$. Form a Weierstrass product over all the poles of $R(\lambda)$:

$$P(\lambda) = \prod_{\lambda_j \in \mathcal{R}, \Im \lambda_j \neq 0} E\left(\frac{\lambda}{\lambda_j}, [m]\right),\ E(z,p) = (1-z)\exp\left(z + \cdots + \frac{z^p}{p}\right).$$

The assumption on Φ ensures that this converges. Then

$$s(\lambda) = e^{g(\lambda)} \frac{P(-\lambda)}{P(\lambda)}$$

for some polynomial g of order at most $[m]$ (Zworski). This relationship can be used to prove (2), [24]. [23]

11. "Inverse" results

I am aware of relatively few "inverse" results for resonances or the scattering phase.[24]

For one-dimensional potential scattering, with $V \in L^\infty_{\text{comp}}(\mathbb{R};\mathbb{R})$ even and real-valued, there is the following result due to Zworski:

- If 0 is not a resonance of V, then the resonances determine V uniquely.
- If 0 is a resonance of V, then there is precisely one even potential $V_1 \in L^\infty_{\text{comp}}(\mathbb{R})$, $V_1 \neq V$, with the same set of scattering poles as V.

Further, related results can be found in [2, 5].

Hassell and Zelditch considered the Dirichlet Laplacian on the connected domain $\mathbb{R}^2 \setminus \mathcal{O}$, with $\partial\mathcal{O}$ smooth. They showed that each class of isophasal domains is sequentially compact in the C^∞- topology. Zelditch [21] showed that the resonances

[22] For example, Selberg showed that for $\Gamma \subset PSL(2;\mathbb{Z})$ that the eigenvalue counting function for $\mathbb{H}^2/\Gamma$ satisfies Weyl's law. In general, the eigenvalue counting function may not satisfy Weyl's law; see for example [4, 10]. In fact, Deshouillers, Iwaniec, Phillips and Sarnak conjecture that the generic Riemann surface with cusps has only a finite discrete spectrum.

[23] This shows that in odd dimensions, the resonances determine the scattering phase up to finitely many constants.

[24] There are many inverse results starting with the scattering matrix.

for the Dirichlet Laplacian in the exterior of a mirror symmetric configuration of analytic domains in $\mathbb{R}^2$ (satisfying certain conditions) determine the obstacle. In the "opposite" direction, Brooks and Perry constructed manifolds X_1, X_2 with the same scattering phase such that $X_i \setminus K_i$ is isometric to $\mathbb{R}^2 \setminus B(R_i)$, where K_i is compact, but X_1 is not isometric to X_2. It is worth noting that having the same scattering phase implies having the same resonances with nonzero imaginary part. In dimension at least 9, Gordon and Perry have families of examples of compactly supported perturbations of Euclidean metrics with the same scattering poles.

If Γ/H is a hyperbolic surface of finite area, then the resonance set (slightly different than our $\mathcal{R}$) determines Γ/H up to finitely many possibilities (Müller). Non-isometric, non-compact, finite volume hyperbolic surfaces with the same scattering matrix were constructed by Bérard and Zelditch.

We mention several results a bit further afield. Borthwick, Judge, and Perry have some results on the compactness in the C^∞ topology of surfaces with the same eigenvalues and scattering poles, where they consider classes of convex co-compact Riemann surfaces, or certain perturbations. Examples of infinite volume isopolar surfaces are given by Guillopé-Zworski, and by Brooks and coauthors (see appendix to [1]). There are examples of nonisometric infinite volume, hyperbolic 3 manifolds, with the same scattering poles and conformally equivalent boundaries (Brooks-Gornet-Perry).

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