Inverse Spectral Theory I: Wave Invariants on Riemannian manifolds (mainly without boundary)

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Goal of Lecture I

In this lecture, we plan to

- Define the wave invariants of the Laplacian;
- Explain how to calculate them from a parametrix on a manifold without boundary.
- Explain how to calculate them from a Birkhoff normal form of the Laplacian around a closed geodesic on a manifold without boundary.

The second lecture will discuss manifolds with boundary.

Laplacian on a (compact) Riemannian manifold

Let (M, g) be a compact Riemannian manifold. In local coordinates x_j we write:

- $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$;
- $[g^{ij}]$ is the inverse matrix to $[g_{ij}]$;
- $g = \det[g_{ij}]$.

The Laplacian of (M, g) is the operator:

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} g^{ij} g \frac{\partial}{\partial x_j}.$$

On a compact manifold, Δ has a discrete spectrum

(1) $$\Delta \varphi_j = \lambda_j^2 \varphi_j, \qquad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}$$

of eigenvalues and eigenfunctions.

Spectral invariants and inverse spectral theory

A spectral invariant is any function of the eigenvalues $\{\lambda_j^2\}$. The inverse spectral problem is to determine information about (M, g) from $\{\lambda_j\}$.

The basic strategy:

- Define a lot of spectral invariants;
- Compute them as explicitly as possible;
- Determine information about (M, g) from the spectral invariants.

Some basic spectral invariants

The simplest spectral invariants come from taking traces

$$TrF_s(\Delta) = \sum_j F_s(\lambda_j),$$

where $F_s(\lambda)$ is an analytic (in s) family of functions.

Examples:

- Heat trace: $Tre^{-t\Delta} = \sum_j e^{-t\lambda_j^2}$;
- Zeta function: $Tr\Delta^{-s} = \sum_j \lambda_j^{-2s}$;
- Wave trace $Tre^{it\sqrt{\Delta}} = \sum_j e^{it\lambda_j}$;
- Determinant: $Z'(0)$.

Singularities of traces

As a general principle, the *singularities* of $TrF_s(\Delta)$ are the simplest spectral invariants. E.g.:

- Asymptotics of $Tre^{-t\Delta}$ as $t \to 0$;
- Poles and residues of $Tr\Delta^{-s}$;
- Singularities of $Tre^{it\sqrt{\Delta}}$ at $t = 0$ or lengths of closed geodesics.

All of these singularities are examples of *non-commutative residues*. They are given by integrals of local geometric invariants of (M, g).

Special values are hard to study but can also be valuable. The most famous is $\det \Delta = Z'(0)$.

Wave group on a Riemannian manifold

We will focus on the wave invariants.

The wave group of (M, g) is the unitary group

$$U(t) = e^{it\sqrt{\Delta}},$$

where $\sqrt{\Delta}$ is defined by the spectral theorem:

$$\sqrt{\Delta} = \sum_j \lambda_j \varphi_j(x)\varphi_j(y).$$

The (Schwartz) kernel of $U(t)$ has the eigenfunction expansion

$$U(t)(x, y) = \sum_j e^{it\lambda_j} \varphi_j(x)\varphi_j(y) \tag{2}$$

$U(t)$ is the solution operator of the 'half' wave equation:

$$\begin{cases} (\frac{\partial}{\partial t} - \sqrt{\Delta})u = 0 \\ u|_{t=0} = f \end{cases} \tag{3}$$

Even/Odd parts of $U(t)$

Closely related but somewhat simpler is the even part of the wave kernel, $\cos t\sqrt{\Delta}$ which solves the initial value problem

$$\begin{cases} (\frac{\partial}{\partial t}^2 - \Delta)u = 0 & \\ u|_{t=0} = f & \frac{\partial}{\partial t} u|_{t=0} = 0 \end{cases} \tag{4}$$

Similar, the odd part of the wave kernel, $\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}$ is the operator solving

$$\begin{cases} (\frac{\partial}{\partial t}^2 - \Delta)u = 0 & \\ u|_{t=0} = 0 & \frac{\partial}{\partial t} u|_{t=0} = g \end{cases} \tag{5}$$

These kernels only really involve Δ.

Riesz-Hadamard parametrices when $\partial M = \emptyset$

For small $t < \mathrm{inj}(M, g)$ one may approximate $U(t)$ by a geometrically constructed kernel:

$$U(t)(x, y) \sim \int_o^\infty e^{i\theta(r^2(x,y)-t^2)} \sum_{k=0}^\infty W_k(x, y)\theta^{\frac{d-3}{2}-k} d\theta$$

where θ^r is regularized at $\theta = 0$, and where

- $W_o(x,y) = \Theta^{-\frac{1}{2}}(x,y)$, where $\Theta(x,y)$ = volume density in normal coordinates at x;
- $r = r(x,y)$ denote the distance function of M;
- $W_j, j \geq 1$ is obtained by solving a transport equations along the geodesic from x to y.

The meaning of $\sim$

Since $\int_o^\infty e^{i\theta\rho}\theta^{\frac{d-3}{2}-k}d\theta = C_{dk}(\rho + i0)^{-\frac{d-3}{2}+k+1}$, we may express $U(t)$ as

$$\sum_{k=0}^{M} W_k(x,y)(r^2(x,y) - t^2 + i0)^{-\frac{d-3}{2}+k+1}$$

plus an error which has the smoothness of $(r^2(x,y) - t^2 + i0)^{-\frac{d-3}{2}+M+1}$.

Thus, $\sim$ is an approximation in terms of smoothness on $\mathbb{R} \times M \times M$

Using the group property of $U(t)$, the short time parametrix determines the wave kernel for all times. It shows that for fixed (x,t) the kernel $U(t)(x,y)$ is singular along the distance sphere $S_t(x)$ of radius t centered at x; hence, singularities propagate along geodesics.

Geometry of geodesics: $\partial M = \emptyset$

We are interested in geodesics, especially closed geodesics. They may be viewed in terms of critical points of the length functional $\mathcal{L}(\gamma)$ on the H^1 loopspace $\Lambda(M)$, or as closed orbits of the Hamilton flow of the length function $|\xi|_g$ on T^*M. We introduce the notation:

- $\mathcal{G}(M) \subset \Lambda(M)$ = closed geodesics: $\delta\mathcal{L}(\gamma) = 0$;
- $G^t : T^*M - 0 \to T^*M - 0$ = the geodesic flow, i.e. the Hamilton flow of $|\xi|_g$.
- γ = denote a closed geodesic, i.e. a closed orbit of G^t in S^*M.
- $Lsp(M,g)$ = length spectrum = set of lengths of closed geodesics.

Bumpy metrics when $\partial M = \emptyset$

We call a metric g on (M,g) with $\partial M = \emptyset$ *non-degenerate* if the energy functional E on $\Lambda(M)$ is Bott-Morse, i.e. $\mathcal{G}(M)$ is a smooth submanifold of $\Lambda(M)$, and $T_c\mathcal{G}(M) = \ker J_c$ where $c \in \mathcal{G}(M)$ and J_c is the Jacobi operator (= index form) on $T_c\Lambda(M)$, i.e. $J_c = \nabla^2 + R(\dot{c},\cdot)\dot{c}$; We also say that g is *bumpy* if, for every $c \in \mathcal{G}(M)$, the orbit $S^1(c)$ of c under the S^1-action of constant reparametrization $c(t+s)$ of $c(t)$, is a non-degenerate critical manifold of E.

When $\mathcal{L}$ and E are Bott-Morse, $Lsp(M, g)$ is a discrete set. One can make finite dimensional approximations to $\Lambda(M)$ on the sets $\mathcal{L} \leq L$.

Geometry of geodesics: $\partial M \neq \emptyset$

In this case, the geodesic flow becomes the billiard flow: $G^t(x, \xi)$ proceeds along a geodesic in M until it hits ∂M. If it hits transversally, it reflects according to Snell's law. If the $x \in \partial M$ and ξ is tangent to ∂M, the geodesic can move along ∂M. When ∂M is convex, the only geodesics which touch ∂M tangentially are the geodesics on ∂M.

If ∂M is not convex, there exist geodesics which touch ∂M tangentially but do not stay forever on ∂M. If a geodesic touches ∂M tangentially to first order, it proceeds as if no intersection occurred. If it hits tangentially to second order, one loses uniquess of the flow: it could glide along $\partial\Omega$ or it could proceed in M.

The functionals $\mathcal{L}, E$ are not Morse functions when $\partial M \neq \emptyset$. $Lsp(M, g)$ has accumulation points.

Closed geodesics: $\partial M \neq \emptyset$

There are several types of closed geodesics:

- Periodic reflecting rays which bounce transversally off the boundary. If there are M reflection points, the trajectory is an M-link geodesic polygon which satisfieds Snell's law at every vertex.
- Any closed geodesic on ∂M is a closed geodesic of M.
- In the non-convex case: glancing geodesics which run along $\partial\Omega$ for some (but not all) of the time.

Glancing geodesics or boundary geodesics are accumulation points for M-bounce reflecting orbits, with $M >> 0$ and with many small bounces.

Jacobi fields and Poincare map

We let $\mathcal{J}_\gamma^\perp \otimes \mathbb{C}$ denote the space of complex normal Jacobi fields along γ, a symplectic vector space of (complex) dimension $2n$ ($n = \dim M - 1$) with respect to the Wronskian

$$\omega(X, Y) = g(X, \frac{D}{ds}Y) - g(\frac{D}{ds}X, Y).$$

The linear Poincaré map P_γ is then the linear symplectic map on $\mathcal{J}_\gamma^\perp \otimes \mathbb{C}$ defined by

$$P_\gamma Y(t) = Y(t + L_\gamma).$$

Eigenvalues of the Poincare map

P_γ is symplectic, so its eigenvalues come in three types:

- pairs $e^{\pm i\alpha_j}$;
- pairs $e^{\pm \lambda_j}$;
- 4-tuplets $e^{\pm \mu_j \pm i\nu_j}$.

We say P_γ is *non-degenerate* if $det(I - P_\gamma) \neq 0$. A closed geodesic is called *elliptic* if all of its eigenvalues are of modulus one, *hyperbolic* if they are all real, and *loxodromic* if they all come in quadruples as above.

Singularity trace of wave group: $\partial M = \emptyset$

The trace $\operatorname{Tr} U(t)$ is a tempered distribution in t with singularities at $t \in Lsp(M, g)$.

On a compact, 'bumpy' (M, g) without boundary, $\operatorname{Tr} U(t)$ is a "Lagrangean distribution". It has special singularities. Its singularity expansion has the form

$$\operatorname{Tr} U(t) = e_0(t) + \sum_{L \in Lsp(M,g)} e_L(t)$$

where

$$e_0(t) = a_{0,-n}(t + iO)^{-n} + a_{0,-n+1}(t + iO)^{-n+1} + \cdots$$

is the singularity at $t = 0$ and where

$$e_L(t) = a_{L,-1}(t - L + iO)^{-1} + a_{L,0} \log(t - (L + iO))$$

$$+ a_{L,1}(t - L + iO) \log(t - (L + iO)) + \cdots$$

is the singularity at $t = L$. where $\cdots$ refers to homogeneous terms of ever higher integral degrees.

Singularity trace of wave group: $\partial M \neq \emptyset$

We now must place boundary conditions on Δ. We will consider only Dirichlet or Neumann boundary conditions. In either case,the trace $\operatorname{Tr} U(t)$ is still a tempered distribution in t with singularities at $t \in Lsp(M, g)$, but is no longer Lagrangean since $Lsp(M, g)$ has accumulation points. However, if γ is a periodic transversal reflecting ray, and if $L = L_\gamma$ is isolated in $Lsp(M, g)$, then $\operatorname{Tr} U(t)$ is Lagrangean near $t = L$ and one has the same kind of singularity expansion at $t = L_\gamma$ (Andersson-Melrose):

$$e_L(t) = a_{L,-1}(t - L + iO)^{-1} + a_{L,0} \log(t - (L + iO))$$

$$+ a_{L,1}(t - L + iO) \log(t - (L + iO)) + \cdots$$

Wave trace coefficients at a non-degenerate orbit

The wave coefficients $a_{0,k}$ at $t = 0$ are given by integrals over M of $\int_M P_j(R, \nabla R, ...)\text{dvol}$ of homogeneous curvature polynomials. They are essentially the same as the heat invariants.

The principal wave invariant at $t = L$ in the case $\partial M = \emptyset$ of a non-degenerate closed geodesic is given by

$$a_{L,-1} = \sum_{\gamma : L_\gamma = L} \frac{e^{\frac{i\pi}{4}} L^\#_\gamma}{|\det(I - P_\gamma)|^{\frac{1}{2}}}$$

where $\{\gamma\}$ runs over the set of closed geodesics, and where L_γ, $L^\#_\gamma$, m_γ, resp. P_γ are the length, primitive length, Maslov index and linear Poincaré map of γ.

The same formula is valid if $\partial\Omega \neq \emptyset$ at an M-bounce transversal periodic reflecting ray except that one gets an extra sign, $(-1)^M$ in the Dirichlet case.

Dual asymptotics

A dual definition of the wave invariants is this: Let $\hat{\rho} \in C_0^\infty(\mathbb{R})$ have support in an interval $[L - \epsilon, L + \epsilon]$ with just one $L \in Lsp(M, g)$. Form

$$\rho(\sqrt{\Delta} - \lambda) = \int_{\mathbb{R}} \hat{\rho}(t) e^{-it\lambda} TrU(t).dt$$

Then $\rho(\sqrt{\Delta} - \lambda)$ is a trace-class operator and we can ask for the asymptotics of $Tr\rho(\sqrt{\Delta} - \lambda)$ as $\lambda \to \infty$. Since $TrU(t) = e_L(t)$ plus terms which are smooth in supp$\hat{\rho}$,

$$Tr\rho(\sqrt{\Delta} - \lambda) = \int_{\mathbb{R}} \hat{\rho}(t) e^{-it\lambda} e_L(t).dt$$

mod $\lambda^{-\infty}$. Since

$$\int_{\mathbb{R}} \hat{\rho}(t) e^{-it\lambda} (t - L + iO)^n \log(t - (L + iO)) dt \sim e^{i\lambda L} \lambda^{-n-1},$$

we get

$$Tr\rho(\sqrt{\Delta} - \lambda) \sim e^{i\lambda L}\{a_{L,-1} + a_{L,0}\lambda^{-1} + a_{L,1}\lambda^{-2} + \cdots\}$$

Reference: J.J.Duistermaat and V.Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, Inv.Math. 24 (1975), 39-80.

Methods for computing wave trace coefficients

There are basically two different methods for computing the wave invariants at a closed geodesic γ when $\partial M = \emptyset$:

- (i) Construct a microlocal parametrix

$$E(t,x,y) = \int_{\mathbb{R}^n} e^{i\varphi(t,x,y,\eta)} a(t,x,y,\eta) d\eta$$

  for $U(t)$ near γ. Here, $\varphi(t,x,y,\eta)$ is a homogeneous phase function and a is a symbol. Write $Tr\rho(\sqrt{\Delta}-\lambda)$ as

$$Tr \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_M \hat{\rho}(t) e^{-i\lambda t} e^{i\varphi(t,x,x,\eta)}$$

$$a(t,x,x,\eta) d\eta dt dV(x).$$

  Apply the stationary phase method to calculate the expansion.

- (ii) Compute by putting $U(t)$ into "Birkhoff normal form" around γ;

Methods (cont)

- The parametrix construction is complicated if (M,g) has conjugate points, but is elementary when (M,g) has no conjugate points. One obtains complicated formulae for the wave invariants in terms of curvature and Jacobi fields along γ. To date, these complicated expressions have not led to any concrete inverse results.
- The Birkhoff normal form construction is straightforward whether or not (M,g) has conjugate points and gives concrete formulae for wave invariants. It has led to the determination of analytic surfaces of revolution from their spectra.

Key ideas

- From the wave invariants $a_{\gamma^r,j}$ of single γ and its iterates γ^r one only obtains information about the jet of g near γ. If g is analytic, this gives global information.
- Try to separate out geometric invariants by comparing $a_{\gamma^r,j}$ as r varies.
- When (M,g) involves only one unknown function of one variable, e.g. a surface of revolution or a plane domain, one can hope to determine it from $a_{\gamma^r,j}$ for one γ.

Methods (cont.)

If $M=\Omega\subset\mathbb{R}^n$ is a bounded, piece-wise smooth Euclidean domain, then there is a further method based on the fact that the Dirichlet resolvent can be expressed as a perturbation of the free resolvent of $\mathbb{R}^n$.

We will discuss this in more detail in the second lecture.

For the remainder of this lecture, we will discuss the two methods when $\partial M = \emptyset$. We begin with the parametrix method on a manifold WCP (without conjugate points).

References

Wave equation on manifolds without conjugate points and wave trace invariants:

(i) P. Berard, On the wave equation without conjugate points, Math. Zeit. **155** (1977), 249–276.

(ii) H. Donnelly, On the wave equation asymptotics of a compact negatively curved surface, Inv. Math. **45** (1978) 115-137.

(iii) T. Sunada, Trace formula and heat equation asymptotics for a non-positively curved manifold, Am. J. Math., vol. 104 (1982), 795–812. (iv)

S. Zelditch, Lectures on wave invariants. Spectral theory and geometry (Edinburgh, 1998), 284–328, London Math. Soc. Lecture Note Ser., 273, Cambridge Univ. Press, Cambridge, 1999.

Geometry of (M, g) WCP:

By definition, $\exists!$ geodesic between any two points (x, y) of the universal cover $\tilde{M}$ and the geodesic distance function (squared) is a global smooth function $r^2(x, y)$.

Notation: $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ = universal riemannian cover; $\Gamma = \pi_1(M)$ = deck transformation group; For $\gamma \in \Gamma$, $f_\gamma(x) = r(x, \gamma x)^2$ =displacement function.

On $\tilde{M}$, the wave operator $\tilde{E}$ can be globally constructed (modulo $C^\infty(\mathbb{R} \times M \times M)$) by the Hadamard-Riesz parametrix method. That is, the wave kernel $\tilde{E}(t, x, y) = \cos(t\sqrt{\Delta})$ is given Mod C^∞ by

$$C_o|t| \sum_{j=0}^{\infty} (-1)^j w_j(x, y) \frac{(r^2 - t^2)_-^{j - \frac{d-3}{2} - 2}}{4^j \Gamma(j - \frac{d-3}{2} - 1)}$$

where C_o is a universal constant and where $W_j = \tilde{C}_o e^{-ij\frac{\pi}{2}} 4^{-j} w_j(x, y),$.

The Hadamard-Riesz coefficients W_j are determined inductively by the transport equations

$$\frac{\Theta'}{2\Theta} W_0 + \frac{\partial W_0}{\partial r} = 0$$

$$4ir(x, y)\{(\frac{k+1}{r(x,y)} + \frac{\Theta'}{2\Theta})W_{k+1} + \frac{\partial W_{k+1}}{\partial r}\} = \Delta_y W_k.$$

The solutions are given by:

$$W_0(x, y) = \Theta^{-\frac{1}{2}}(x, y)$$

$$W_{j+1}(x, y) = \Theta^{-\frac{1}{2}}(x, y) \int_0^1 s^k \Theta(x, x_s)^{\frac{1}{2}} \Delta_2 W_j(x, x_s) ds$$

where x_s is the geodesic from x to y parametrized proportionately to arc-length and where Δ_2 operates in the second variable.

Wave kernel on (M, g) WCP:

The wave kernel $E(t, x, y)$ on M is obtained by projecting this kernel from $\tilde{M}$, i.e. by summing over the deck transformation group:

$$E(t, x, y) = \sum_{\gamma \in \Gamma} \tilde{E}(t, x, \gamma \cdot y)$$

$$\equiv C_o |t| \sum_{\gamma \in \Gamma} \sum_{j=0}^{\infty} (-1)^j w_j(x, \gamma y) \frac{(r(x,\gamma y)^2 - t^2)_-^{j - \frac{d-3}{2} - 2}}{4^j \Gamma(j - \frac{d-3}{2} - 1)}$$

Formula for sub-principal wave invariant

The subprincipal wave invariant $a_{\gamma o}$ equals :

$$C^o_{n,o,1} \int_\gamma \frac{w_1(x, \gamma x) d\sigma}{\sqrt{\det \mathrm{Hess}(f\gamma)_\sigma}}$$

$$+ C^o_{n,o,o} \int_\gamma \frac{\mathrm{Hess}(f_\gamma)_\sigma^{-1}(\frac{\partial}{\partial y}, \frac{\partial}{\partial y})[(w_0(x, \gamma x) J_\gamma(y,s)] d\sigma}{\sqrt{\det \mathrm{Hess}(f_\gamma)_\sigma}}$$

for certain universal constants. Here, $J_{[\gamma]}$ is the volume density in certain coordinates.

Using Fermi coordinates along γ one can derive from this a polynomial expression in terms of curvature and Jacobi polynomials along γ. This explains what kind of data goes into $a_{\gamma^r, j}$, but the behaviour in r is not transparent.

Birkhoff normal forms: $\partial M = \emptyset$

A much better method is to construct a quantum Birkhoff normal form of $\sqrt{\Delta}$ around a non-degenerate closed geodesic γ.

Roughly speaking, to put $\sqrt{\Delta}$ into microlocal normal form around γ is to express it as a function $F(D_s, \hat{I}_1, \dots, \hat{I}_n)$ ($n = \dim M - 1$) of the derivative $D_s = \frac{\partial}{i \partial s}$ along γ and of local commuting 'action operators' $\hat{I}_j$ along a transversal to γ. The definition of the $\hat{I}_j$ depends on the spectrum of P_γ.

References when $\partial M = \emptyset$

1. V.Guillemin, Wave trace invariants, Duke Math. J. 83 (1996), 287-352.

2. V. Guillemin, Wave-trace invariants and a theorem of Zelditch, Duke Int.Math.Res.Not. 12 (1993), 303-308.

3. A. Iantchenko, J. Sjostrand and M. Zworski, Birkhoff normal forms in semiclassical inverse problems.

4. S.Zelditch, Wave invariants at elliptic closed geodesics, Geom.Anal.Fun.Anal. 7 (1997), 145-213.

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6. ———-, The inverse spectral problem for surfaces of revolution, J. Diff. Geom. 49 (1998), 207-264.

The model space

The wave invariants $a_{\gamma k}$ associated to a closed geodesic γ depend only on the germ of the metric in a neighborhood of γ. We may identify this neighborhood with the normal bundle N_γ by means of the exponential map. Thus, the model space is the cylinder $S^1_L \times \mathbb{R}^n$, where $S^1_L = \mathbb{R}/L\mathbb{Z}$. We use the coordinates $(s, y) \in S^1_L \times \mathbb{R}$ and dual symplectic coordinates (σ, η) for $T^*(S^1_L \times \mathbb{R})$. We henceforth assume the length L of γ equals 2π.

Action operators

Define Y_j = "multiplication by y_j" and by $D_{y_j} = \frac{\partial}{i\partial y_j}$. The symplectic algebra $sp(n, \mathbb{C})$ is represented by homogeneous quadratic polynomials in Y_j, D_j. A choice of action operators is equivalent to a choice of a maximal abelian subalgebra $\mathcal{I} :=< \hat{I}_1, ..., \hat{I}_n >$ of $sp(n, \mathbb{C})$. The appropriate choice of $\mathcal{I}$ depends on type of closed geodesic γ, specifically on the spectrum of P_γ. For each pair $e^{i\pm\alpha}$ of elliptic eigenvalues, one introduces the elliptic harmonic oscillator $\hat{I}_j := \alpha_j(D^2_{y_j} + y_j^2)$. In the real hyperbolic directions, i.e. in real 2-planes where P_γ has a pair of inverse real eigenvalues $e^{\pm\mu_j}$, one introduces hyperbolic action operators $\hat{I}_j = \mu_j(y_j D_{y_j})$. Similarly in the complex hyperbolic directions.

Birkhoff normal forms

There exists a microlocal conjugation:

$$W\sqrt{\Delta_\psi}W^{-1} \equiv D_s + \frac{1}{L}H_\alpha + \frac{\tilde{p}_1(\hat{I}_1,...,\hat{I}_n)}{D_s}$$

$$+\frac{\tilde{p}_2(\hat{I}_1,...,\hat{I}_n)}{D_s^2} + \cdots + \frac{\tilde{p}_{k+1}(\hat{I}_1,...,\hat{I}_n)}{D_s^{k+1}} + \ldots$$

where the numerators $\tilde{p}_j(\hat{I}_1, ..., \hat{I}_n)$ are polynomials of degree $j+1$ in the variables $\hat{I}_1, ..., \hat{I}_n$. The k-th remainder term lies in the space $\bigoplus_{j=0}^{k+2} O_{2(k+2-j)}\Psi^{1-j}$.

Birkhoff normal form invariants

The coefficients of the polynomials

$$\tilde{p}_j(\hat{I}_1, ..., \hat{I}_n)$$

are the quantum Birkhoff normal form invariants. We denote them by $B_{\gamma jk}$. They have the form $B_{\gamma k} = \int_\gamma I_{\gamma;k}(s;g)ds$ where $I_{\gamma;k}(s;g)$ is a 'homogeneous Fermi–Jacobi–Floquet polynomial' of weight $-k-1$ in the data $\{y_{ij}, \dot{y}_{ij}, D^m_{s,y} g\}$ with $m = (m_1, \ldots, m_{n+1})$ satisfying $|m| \leq 2k+4$ with various constraints.

Notation

Let us define the term 'Fermi-Floquet-Jacobi polynomial'. Contractions of the curvature tensor and its covariant derivatives against Fermi coordinate vector fields and against the Jacobi eigenfields $Y_j, \overline{Y}_j$, are called *Fermi–Jacobi polynomials*. We denote the coefficients of Y_j by y_{jk}. Floquet means that in addition we have powers of the invariants $\beta_j = (1 - e^{i\alpha_j})^{-1}$.

The 'weights' referred to above describe how the various objects scale under $g \to \epsilon^2 g$.

Explicit formulae for the supbrincipal wave invariant

In dimension 2 (where there is only one Floquet invariant β) the residual wave invariant $a_{\gamma 0}$ is given by:

$$a_{\gamma 0} = \frac{a_{\gamma,-1}}{L^\#}[B_{\gamma 0;4}(2\beta^2 - \beta - \tfrac{3}{4}) + B_{\gamma 0;0}]$$

where:
(a) $a_{\gamma,-1}$ is the principal wave invariant;
(b) $L^\#$ is the primitive length of γ; σ is its Morse index; P_γ is its Poincaré map;
(c) $B_{\gamma 0;j}$ has the form:

$$\frac{1}{L^\#}\int_o^{L^\#}[a\,|\dot{Y}|^4 + b_1\,\tau|\dot{Y}\cdot Y|^2 + b_2\,\tau \mathrm{Re}\,(\bar{Y}\dot{Y})^2$$

$$+c\,\tau^2|Y|^4 + d\,\tau_{\nu\nu}|Y|^4 + e\,\delta_{j0}\tau]ds$$

$$+\frac{1}{L^\#}\sum_{0\leq m,n\leq 3; m+n=3} C_{1;mn}\,\frac{\sin((n-m)\alpha)}{|(1-e^{i(m-n)\alpha})|^2}$$

$$\left|\int_o^{L^\#}\tau_\nu(s)\bar{Y}^m\cdot Y^n(s)ds\right|^2$$

$$+\frac{1}{L^\#}\sum_{0\leq m,n\leq 3; m+n=3} C_{2;mn}\ \mathrm{Im}\ \int_o^{L^\#}\tau_\nu(s)$$

$$\bar{Y}^m\cdot Y^n(s)\left[\int_o^s \tau_\nu(t)\bar{Y}^n\cdot Y^m(t)dt\right]ds\}$$

for various universal (computable) coefficients. Here, τ= scalar curv.

How to construct the normal form

To put Δ into BNF around γ:

(i) Re-scale the Laplacian around γ. It then depends on a small parameter h.

(ii) Treat the result as a semi-classical operator and conjugate it to a semi-classical normal form by successive conjugations with semiclassical Fourier integral operators.

(iii) Deduce a homogeneous normal form for Δ (without small parameter).

Re-scaling the Laplacian

We introduce the unitary operators T_h and M_h on $L^2_T(\mathbb{R}^1 \times \mathbb{R}^n)$:

$$\begin{array}{l} T_h f(s,u) := h^{-n/2} f(s, h^{-\frac{1}{2}} u), \\ M_h f(s,u) := e^{\frac{i}{hL} s} f(s,y) \end{array} .$$

We have: $T_h^* D_{u_j} T_h = h^{-\frac{1}{2}} D_{u_j}$, $T_h^* u_i T_h = h^{\frac{1}{2}} u_i$, $M_h^* D_s M_h = ((hL)^{-1} + D_s)$. All commute. The *re-scaling* of Δ is:

$$\Delta_h := T_h^* M_h^* \Delta T_h M_h$$

Re-scaled Laplacian

We put the (1/2-density) Laplacian in Fermi normal coordinates

$$-\Delta = J^{-1/2} \partial_s g^{oo} J \partial_s J^{-1/2}$$

$$+ \sum_{ij=1}^n J^{-1/2} \partial_{u_i} g^{ij} J \partial_{u_j} J^{-1/2}$$

$$\equiv g^{oo} \partial_s^2 + \Gamma^o \partial_s + \sum_{ij=1}^n g^{ij} \partial_{u_i} \partial_{u_j}$$

$$+ \sum_{i=1}^n \Gamma^i \partial_{u_i} + \sigma_o.$$

We then have:

$$-\Delta_h = -(hL)^{-2} g^{oo}_{[h]} + 2i(hL)^{-1} g^{oo}_{[h]} \partial_s$$

$$+i(hL)^{-1}\Gamma^o_{[h]} + h^{-1}(\sum_{ij=1}^n g^{ij}_{[h]} \partial_{u_i}\partial_{u_j})$$

$$+h^{-\frac{1}{2}}(\sum_{i=1}^n \Gamma^i_{[h]}\partial_{u_i}) + (\sigma)_{[h]}.$$

the subscript $[h]$ indicating to dilate the coefficients of the operator in the form, $f_h(s,u) := f(s, h^{\frac{1}{2}}u)$.

Expanding the coefficients in Taylor series at $h = 0$, we obtain the asymptotic expansion

$$\Delta_h \sim \sum_{m=0}^{\infty} h^{(-2+m/2)} \mathcal{L}_{2-m/2} \tag{2.12}$$

where $\mathcal{L}_2 = L^{-2}$, $\mathcal{L}_{3/2} = 0$ and where

$$\mathcal{L}_1 = [i\frac{\partial}{\partial s} + \frac{1}{2}\{\sum_{j=1}^n \partial^2_{u_j} - \sum_{ij=1}^n K_{ij}(s)u_i u_j\}].$$

We conjugate by an operator in the metaplectic group to get

$$\mathcal{D}_h = \mu(\mathcal{A}^*_L)^{-1}\Delta_h \mu(\mathcal{A}^*_L)$$

which has the asymptotic expansion

$$\mathcal{D}_h \sim \sum_{m=o}^{\infty} h^{(-2+\frac{m}{2})}\mathcal{D}_{2-\frac{m}{2}} \tag{2.15}$$

with $\mathcal{D}_2 = I, \mathcal{D}_{\frac{3}{2}} = 0, \mathcal{D}_1 = D_s$. This simplifies $\mathcal{L}_1$.

We now conjugate inductively with semiclassical pseudodifferential operators of the form:

$$W_{h\frac{k}{2}} := exp(ih^{\frac{k}{2}} Q_{\frac{k}{2}}).$$

and with $h^{\frac{k}{2}} Q_{\frac{k}{2}} \in h^{\frac{k}{2}}\mathbb{C}^\infty(S^1_L)\otimes\mathcal{E}^{k+2}$ of total order 1. We stary by constructing $Q_{\frac{1}{2}}(s,x,D_x)$ such that

$$e^{-ih^{\frac{1}{2}}Q_{\frac{1}{2}}}\mathcal{D}_h e^{ih^{\frac{1}{2}}Q_{\frac{1}{2}}}|_o = [-h^{-2}L^{-2}$$

$$+2h^{-1}L^{-1}\mathcal{D} + \mathcal{D}^{\frac{1}{2}}_o + \ldots]|_o$$

where the dots $\ldots$ indicate higher powers in h. The operator $Q_{\frac{1}{2}}$ then must satisfy the commutation relation

$$\{[L^{-1}\mathcal{D}, Q_{\frac{1}{2}}] + \mathcal{D}_{\frac{1}{2}}\}|_o = 0.$$

that is,

$$L^{-1}\partial_s\{\mu(r_\alpha)^* Q_{\frac{1}{2}}\mu(r_\alpha)\}|_o = -i\{\mathcal{D}_{\frac{1}{2}}\}|_o$$

To solve, we rewrite the equation in terms of complete Weyl symbols:

$$L^{-1}\partial_s\tilde{Q}_{\frac{1}{2}}(s,x,\xi) = -i\mathcal{D}_{\frac{1}{2}}|_o(s,x,\xi)$$

with

$$\tilde{Q}_{\frac{1}{2}}(s+L,x,\xi) = \tilde{Q}_{\frac{1}{2}}(s,r_\alpha(L)(x,\xi)).$$

We solve with the Weyl symbol

$$\tilde{Q}_{\frac{1}{2}}(s,x,\xi) = \tilde{Q}_{\frac{1}{2}}(0,x,\xi) - i\int_0^s \mathcal{D}_{\frac{1}{2}}|_o(u,x,\xi)du$$

where $\tilde{Q}_{\frac{1}{2}}(0,x,\xi)$ is determined by the consistency condition (after changing to complex coordinates)

$$\tilde{Q}_{\frac{1}{2}}(0,e^{i\alpha}z,e^{-i\alpha}\bar{z}) - \tilde{Q}_{\frac{1}{2}}(0,z,\bar{z})$$

$$= \int_0^L \mathcal{D}_{\frac{1}{2}}|_o(u,z,\bar{z})du$$

Since $\mathcal{D}_{\frac{1}{2}}(u,z,\bar{z})$ is a polynomial of degree 3, we can solve with

$$\tilde{Q}_{\frac{1}{2}}(s,z,\bar{z}) = \sum_{|m|+|n|\leq 3} q_{\frac{1}{2};mn}(s)z^m\bar{z}^n$$

and

$$\mathcal{D}_{\frac{1}{2}}|_o(s,z,\bar{z})du = \sum_{|m|+|n|\leq 3} d_{\frac{1}{2};mn}(s)z^m\bar{z}^n.$$

The consistency equation is:

$$\sum\nolimits_{|m|+|n|\leq 3}(1-e^{(m-n)\alpha})q_{\frac{1}{2};mn}(0)z^m\bar{z}^n$$

$$= -iL^2\sum\nolimits_{|m|+|n|\leq 3}\bar{d}_{\frac{1}{2};mn}z^m\bar{z}^n.$$

Since there are no terms with $m = n$ in this (odd-index) equation, and since the α_j's are independent of π over $\mathbb{Z}$, there is no obstruction to the solution.

The even steps behave differently from the odd ones. The normal form comes from the even steps.

We seek an element $\tilde{Q}_1(s, x, D_x)$ and an element $f_o(I_1, ..., I_n) \in \mathcal{A}$ so that

$$\mathcal{D}^1_h := e^{-ih\tilde{Q}_1}\mathcal{D}^{\frac{1}{2}}e^{ih\tilde{Q}_1}$$

$$= h^{-2}L^{-2} + h^{-1}L^{-1}D_s + h^{-\frac{1}{2}}\mathcal{D}^{\frac{1}{2}}_{\frac{1}{2}}$$

$$+\mathcal{D}^1_o(s, D_s, x, D_x) + \dots$$

with

$$\mathcal{D}^1_o(s, D_s, x, D_x)|_o = f_o(I_1, ..., I_n).$$

We get the equation

$$\{[D_s, \tilde{Q}_1] + \mathcal{D}^{\frac{1}{2}}_o\}|_o = f_o(I_1, ..., I_n) \tag{2.35a}$$

or equivalently

$$\partial_s\tilde{Q}_1|_o = \{-\mathcal{D}^{\frac{1}{2}}_o + f_o(I_1, ..., I_n)\}|_o.$$

Now there is an obstruction to solving, which we remove by putting

$$f_o(I_1, ..., I_n) = \frac{1}{L}\int_o^L\int_{T^n} V_t^*\mathcal{D}^{\frac{1}{2}}_o|_o V_t dtds \tag{2.37}$$

where $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is the n-torus, where $t \in T^n$ and where

$$V(t_1, ..., t_n) := expit_1I_1 \cdot \dots \cdot expit_nI_n.$$

To solve, we rewrite as:

$$\partial_s\tilde{Q}_1(s, z, \bar{z}) = \{\mathcal{D}^{\frac{1}{2}}_o|_o(s, z, \bar{z})$$

$$-f_o(|z_1|^2, \dots, |z_n|^2)\}$$

or equivalently

$$\tilde{Q}_1(s, z, \bar{z}) - \tilde{Q}_1(0, z, \bar{z})$$

$$= \textstyle\int_0^s[\mathcal{D}^{\frac{1}{2}}_o|_o(u, z, \bar{z}) - f_o(|z_1|^2, \dots, |z_n|^2)]du$$

and solve simeltaneously for $\tilde{Q}_1$ and f_o. There is a consistency condition determining $\tilde{Q}_1$. We write:

$$\tilde{Q}_1(s, z, \bar{z}) = \textstyle\sum_{|m|+|n|\leq 4} q_{1;mn}(s)z^m\bar{z}^n,$$
$$f_o(|z_1|^2, \dots, |z_n|^2) = \textstyle\sum_{|k|\leq 2} c_{ok}|z|^{2k}$$

We also put

$$\mathcal{D}_o^{\frac{1}{2}}|_o(s,z,\bar{z})du := \sum_{|m|+|n|\leq 4} d_{o;mn}^{\frac{1}{2}}(s) z^m \bar{z}^n, \qquad \bar{d}_{o;mn}^{\frac{1}{2}} := \frac{1}{L}\int_o^L d_{o;mn}^{\frac{1}{2}}(s)ds$$

As above, we can solve for the off-diagonal coefficients,

$$q_{1;mn}(0) = -iL^2(1 - e^{i(m-n)\alpha})^{-1}\bar{d}_{o;mn}^{\frac{1}{2}} \tag{2.43a}$$

and must set the diagonal coefficients equal to zero. The coefficients c_{ok} are then determined by

$$c_{ok} = \bar{d}_{1;kk}^{\frac{1}{2}}. \tag{2.43b}$$

Final remarks

The formulae for the subprincipal normal form coefficients $B_{\gamma^r 0;j}$ are already very complicated and become more so as we go deeper into the normal form expansion. But on a surface of revolution with only one local maximum distnce from the axis, one can calculate the coefficients when γ is the invariant geodesic. One gets the full Taylor expansion of the profile curve at the maximum. A related method can find an analytic plane domain with a symmetry.