Generalized coclass trees

Jack Schmidt

University of Kentucky

Zassenhaus 2007-03-17

Overview

- I am interested in a merger of the methods of finite p-groups and finite simple groups to understand perfect groups
- In this talk I describe some of the success and difficulties in applying the coclass classification of p-groups to perfect groups
- To continue the work of Holt & Plesken, more examples needed
- To compute examples, new comptuational group theory was needed (described two weeks ago), and now I present the results of that theory

Background in p-groups

- The coclass classification understands an infinite family of p-groups as approximate quotients of a single profinite group
- The groups naturally organize into a tree, and the profinite group is the limit of the "trunk" or "main-line" of the tree
- The error in the approximation are called the branches of the tree, and the perfect approximations are called the trunk
- If G is a pro-p-group (or p-group), and G_n the nth term of the lower central series, then $\sup(\log_p([G : G_n]) - n)$ is the coclass
- The graph has nodes finite p-groups and edges from G/G_n to G/G_{n+1} if $[G_n : G_{n+1}] = p$

Background in perfect groups

- Small perfect groups (order less than 10^{10}) are repeated downward extensions of direct products of simple groups by normal p-subgroups
- Constructing these extensions required the development of new methods in computational group theory (described two weeks ago)
- I give some examples of how these groups are gathered into families, including calculations for groups, sometimes up to order $120 \cdot 5^{21} \approx 5.7\text{E}16$

Origin of p-coclass

- Definition due to Holt & Plesken, 1993
- p-groups poorly understood by order after years of effort
- p-groups well understood by coclass
- Holt & Plesken studied perfect groups over a series of papers and a book, but by chief length of the p-core
- They wanted a better understanding, and hoped p-coclass would be the correct method

The definition itself

- Fix a prime p, for a profinite group G define G_n to be the nth term in the lower central series of $O_p(G)$
- For a profinite group G,

$$\mathrm{Coclass}_p(G) = \sup(\ell(O_p(G)/G_n) - n)$$

where $\ell(H)$ denotes the chief length, the length of a maximal chain of normal subgroups

- The coclass graph has nodes all groups and edges from G/G_n to G/G_{n+1} if G_n/G_{n+1} is a simple G-module

Basic results of Holt-Plesken

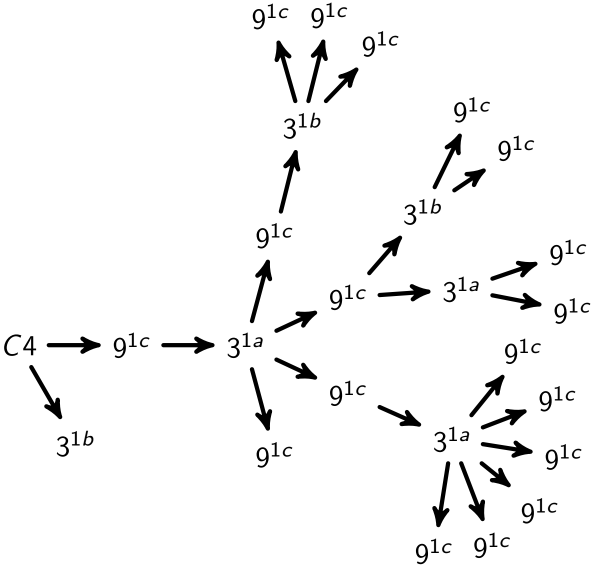
- Once G/G_n has the same coclass as G, G_n has a unique G-chief series
- If $G_1 = O_p(G)$ is soluble, then $G_1/t(G_1)$ is a p-adic space group, where $t(G_1)$ is the torsion subgroup
- ... and for some n, G acts p-uniserially on G_n, a $\hat{\mathbb{Z}}_p[G/G_n]$-lattice
- However, soluble $O_p(G)$ seems rare

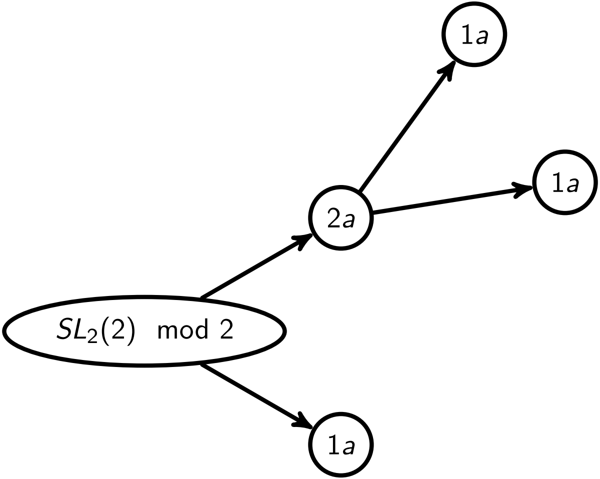
More examples needed

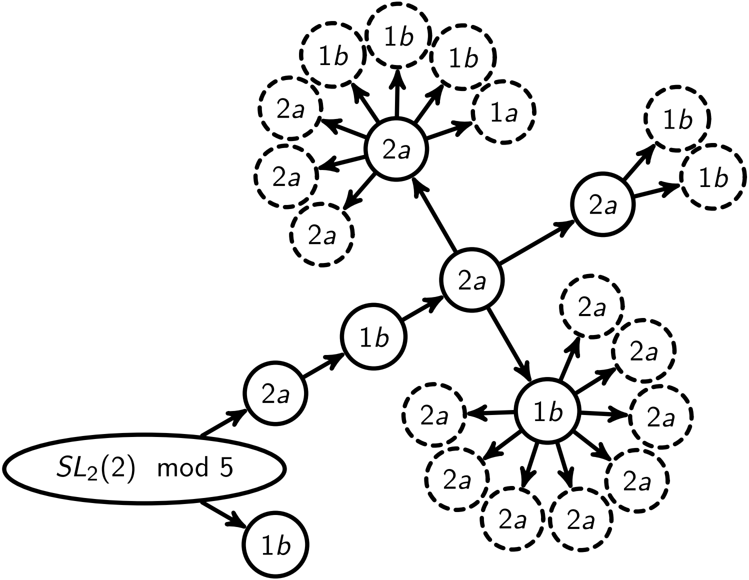
- If there is some regular behaviour to describe, it will be recognized in special cases
- Some infinite families already understood, but given an arbitrary coclass 0 group, how to fit it into a family?
- Holt & Plesken describe a very large family of examples coming from the natural representation of groups of Lie type over $\mathfrak{p}$-adic integers
- Easy examples are $\varprojlim \mathrm{SL}_2(\mathbb{Z}/p^i\mathbb{Z})$ which have $G/O_p(G) = SL(2,p)$ and lower central factors all the irreducible module p^3

Some imperfect examples

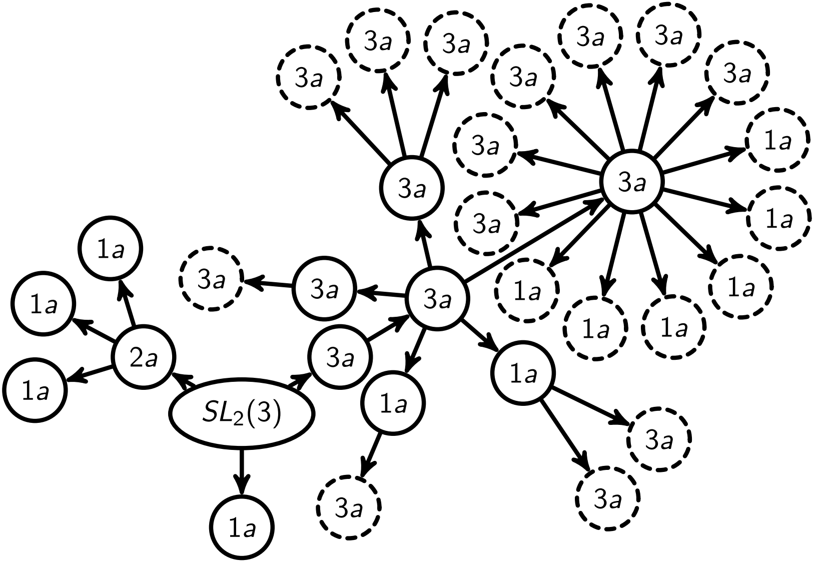
- Even small groups are interesting; $G/O_p(G) = 1$ already covers the coclass classification of p-groups!
- Coclass trees with cyclic roots can be calculated somewhat abstractly ($C_n \mod p$ calculations mostly depend on $p \mod n$)
- Already Sym(3)=SL(2,2) is interesting; its theory mod p just depends on $p \mod 3$, but $p = 2$ is very different from the other 2 mod 3
- Even SL(2,3) has interesting properties, especially compared to perfect SL(2,p)


9^{1c} 9^{1c} 9^{1c}
3^{1b}
9^{1c} 9^{1c}
3^{1b}
9^{1c}
9^{1c} 3^{1a} 9^{1c}
9^{1c}
$C4$ 9^{1c} 3^{1a}
9^{1c} 9^{1c}
3^{1b} 9^{1c} 3^{1a} 9^{1c}
9^{1c}
9^{1c} 9^{1c}


1a
1a
2a
SL2(2) mod 2
1a

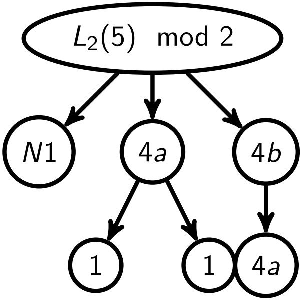

1b
1b
1b
2a
1a
2a
2a
1b
2a
1b
2a
2a
2a
2a
2a
1b
2a
2a
2a
2a
2a
2a
2a
2a
$SL_2(2)$ mod 5
1b

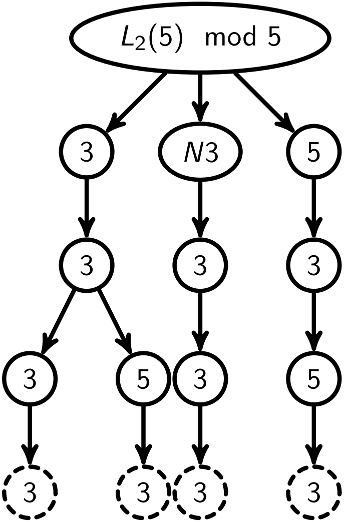
$SL_2(3)$ mod 3


3a
3a
3a
3a
3a
3a
3a
3a
3a
1a
3a
3a
1a
1a
3a
1a
1a
1a
1a
3a
3a
3a
1a
1a
2a
3a
1a
1a
$SL_2(3)$
3a
3a
3a
1a

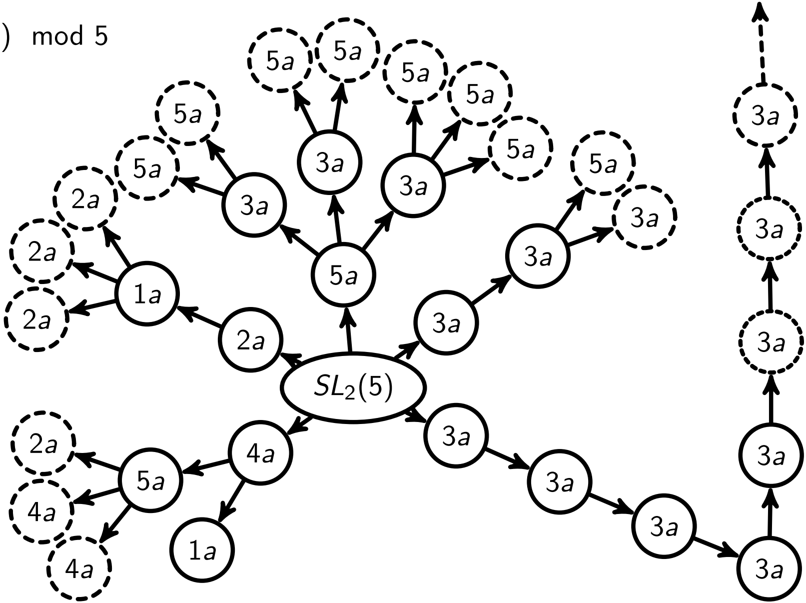
Perfect examples

- I am working out all the details for $SL(2, \mathbb{Z}/p^2\mathbb{Z})$ mod p whose coclass graph is precisely a "main-line" with no branches or twigs
- However starting at SL(2,p) the situation is much messier and appears full of branches, twigs, or even multiple main-lines

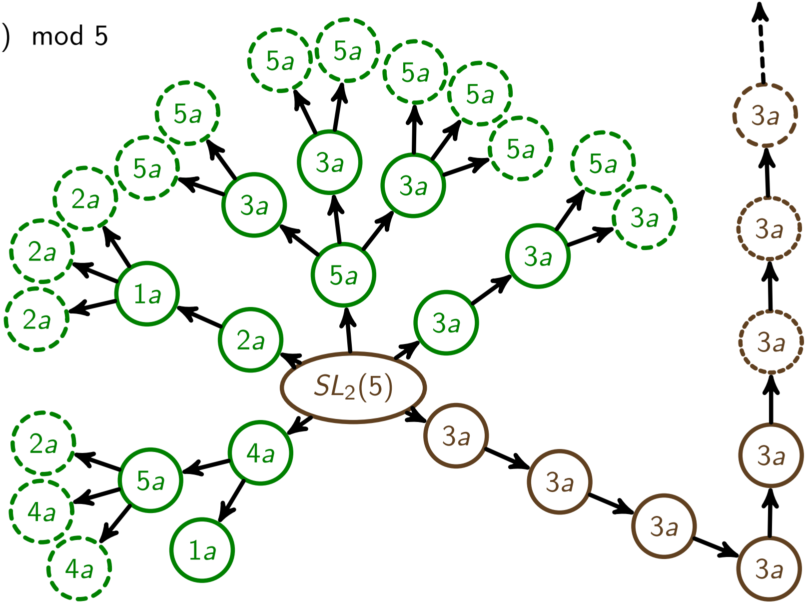

$L_2(5)$ mod 2
$N1$
$4a$
$4b$
1
1
$4a$

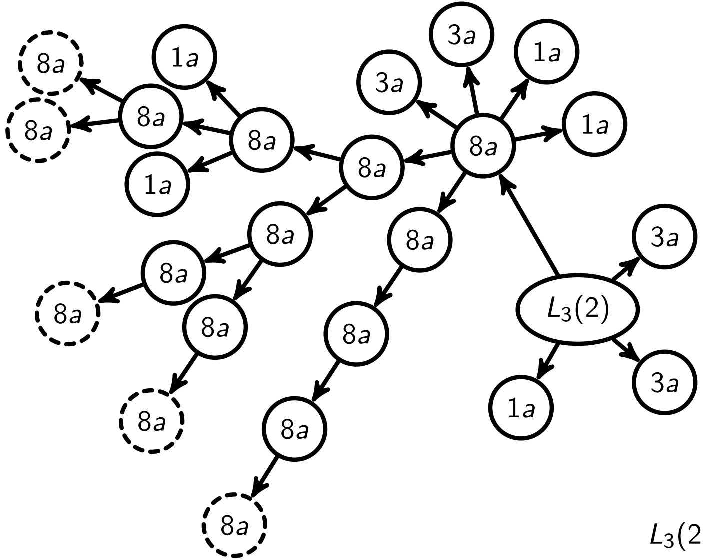

$L_2(5)$ mod 5
3
$N3$
5
3
3
3
3
5
3
5
3
3
3
3

$SL_2(5)$ mod 5

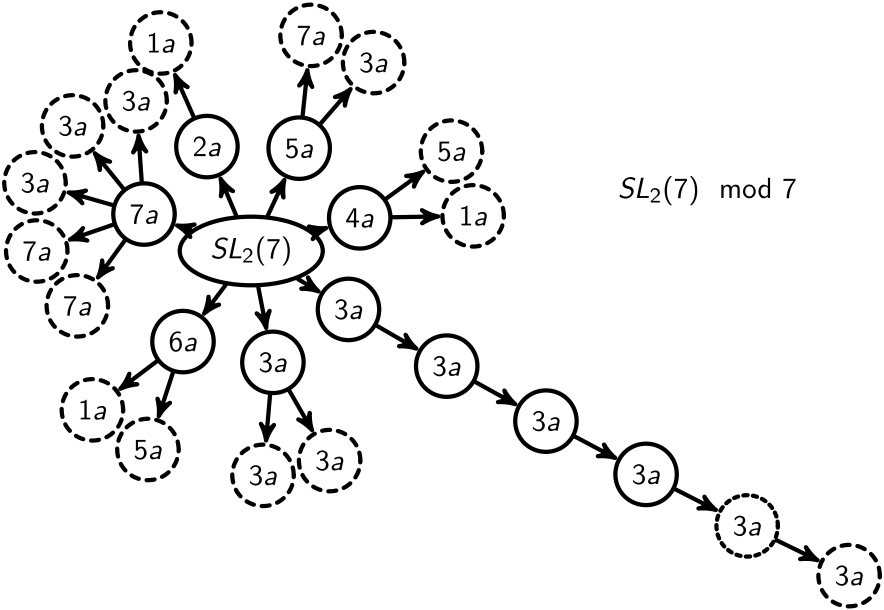

$SL_2(5)$
2a
1a
2a
2a
2a
5a
5a
3a
5a
3a
5a
5a
3a
5a
5a
5a
3a
3a
5a
3a
4a
5a
2a
4a
4a
1a
3a
3a
3a
3a
3a
3a
3a
3a

$SL_2(5)$ mod 5

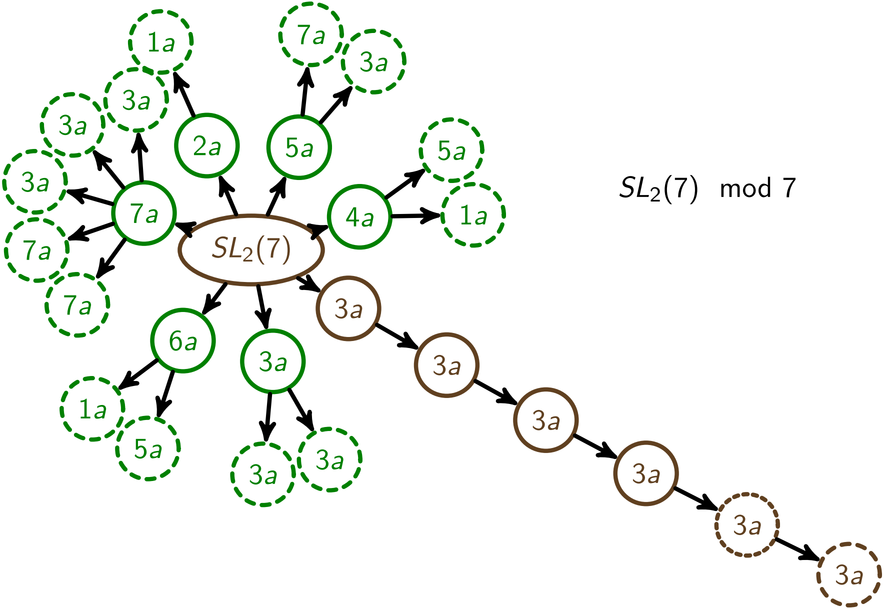

$SL_2(5)$
2a
1a
2a
2a
2a
5a
4a
2a
4a
4a
1a
5a
3a
5a
5a
3a
5a
5a
3a
5a
5a
5a
3a
3a
5a
3a
3a
3a
3a
3a
3a
3a
3a
3a
3a


8a
8a
1a
8a
8a
1a
3a
3a
1a
1a
8a
8a
8a
8a
8a
8a
8a
8a
8a
8a
8a
8a
8a
8a
$L_3(2)$
3a
3a
1a

$L_3(2)$ mod 2


$SL_2(7)$
7a
2a
5a
4a
6a
3a
1a
3a
7a
3a
5a
1a
3a
3a
3a
7a
7a
1a
5a
3a
3a
3a
3a
3a
3a
3a
3a

$SL_2(7)$ mod 7


1a
7a
3a
3a
3a
2a
5a
5a
3a
7a
4a
1a
SL2(7)
7a
3a
7a
6a
3a
3a
3a
1a
3a
5a
3a
3a
3a
3a
3a

$SL_2(7) \mod 7$