Generalized coclass trees

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Overview

- I am interested in a merger of the methods of finite $p$-groups and finite simple groups to understand perfect groups.

- In this talk I describe some of the success and difficulties in applying the coclass classification of $p$-groups to perfect groups.

- To continue the work of Holt & Plesken, more examples needed.

- To compute examples, new computational group theory was needed (described two weeks ago), and now I present the results of that theory.
Background in $p$-groups

- The coclass classification understands an infinite family of $p$-groups as approximate quotients of a single profinite group.

- The groups naturally organize into a tree, and the profinite group is the limit of the “trunk” or “main-line” of the tree.

- The error in the approximation are called the branches of the tree, and the perfect approximations are called the trunk.

- If $G$ is a pro-$p$-group (or $p$-group), and $G_n$ the $n$th term of the lower central series, then $\sup(\log_p([G : G_n]) - n)$ is the coclass.

- The graph has nodes finite $p$-groups and edges from $G/G_n$ to $G/G_{n+1}$ if $[G_n : G_{n+1}] = p$. 
Background in perfect groups

- Small perfect groups (order less than $10^{10}$) are repeated downward extensions of direct products of simple groups by normal $p$-subgroups

- Constructing these extensions required the development of new methods in computational group theory (described two weeks ago)

- I give some examples of how these groups are gathered into families, including calculations for groups, sometimes up to order $120 \cdot 5^{21} \approx 5.7E16$
Origin of $p$-coclass

- Definition due to Holt & Plesken, 1993

- $p$-groups poorly understood by order after years of effort

- $p$-groups well understood by coclass

- Holt & Plesken studied perfect groups over a series of papers and a book, but by chief length of the $p$-core

- They wanted a better understanding, and hoped $p$-coclass would be the correct method
The definition itself

- Fix a prime $p$, for a profinite group $G$ define $G_n$ to be the $n$th term in the lower central series of $O_p(G)$

- For a profinite group $G$, 

$$\text{Coclass}_p(G) = \sup(\ell(O_p(G)/G_n) - n)$$

where $\ell(H)$ denotes the chief length, the length of a maximal chain of normal subgroups

- The coclass graph has nodes all groups and edges from $G/G_n$ to $G/G_{n+1}$ if $G_n/G_{n+1}$ is a simple $G$-module
Basic results of Holt-Plesken

- Once $G/G_n$ has the same coclass as $G$, $G_n$ has a unique $G$-chief series

- If $G_1 = O_p(G)$ is soluble, then $G_1/t(G_1)$ is a $p$-adic space group, where $t(G_1)$ is the torsion subgroup

- ... and for some $n$, $G$ acts $p$-uniserially on $G_n$, a $\hat{\mathbb{Z}}_p[G/G_n]$-lattice

- However, soluble $O_p(G)$ seems rare
More examples needed

- If there is some regular behaviour to describe, it will be recognized in special cases.

- Some infinite families already understood, but given an arbitrary coclass 0 group, how to fit it into a family?

- Holt & Plesken describe a very large family of examples coming from the natural representation of groups of Lie type over $p$-adic integers.

- Easy examples are $\lim_{\leftarrow} \text{SL}_2(\mathbb{Z}/p^i\mathbb{Z})$ which have $G/O_p(G) = \text{SL}(2, p)$ and lower central factors all the irreducible module $p^3$. 
Some imperfect examples

- Even small groups are interesting; $G/O_p(G) = 1$ already covers the coclass classification of $p$-groups!

- Coclass trees with cyclic roots can be calculated somewhat abstractly ($C_n \mod p$ calculations mostly depend on $p \mod n$)

- Already $\text{Sym}(3) = \text{SL}(2,2)$ is interesting; its theory mod $p$ just depends on $p \mod 3$, but $p = 2$ is very different from the other $2 \mod 3$

- Even $\text{SL}(2,3)$ has interesting properties, especially compared to perfect $\text{SL}(2,p)$
Generalized coclass trees

Examples

Imperfect examples
Generalized coclass trees

Examples

Imperfect examples

$SL_2(2) \mod 2$
Generalized coclass trees
Examples
Imperfect examples

$SL_2(2) \mod 5$
$SL_2(3) \mod 3$
I am working out all the details for $SL(2, \mathbb{Z}/p^2\mathbb{Z}) \mod p$ whose coclass graph is precisely a “main-line” with no branches or twigs.

However starting at $SL(2,p)$ the situation is much messier and appears full of branches, twigs, or even multiple main-lines.
**Generalized coclass trees**

**Examples**

**Perfect examples**

$L_2(5) \mod 2$

- $N_1$
- $4a$
- $4b$
- $1$
- $1$
- $4a$

$L_2(5) \mod 5$

- $3$
- $N_3$
- $5$
- $3$
- $5$
- $3$
- $3$
- $3$
- $3$
- $3$
- $3$

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$SL_2(5) \mod 5$
$SL_2(5) \mod 5$
Generalized coclass trees

Examples

Perfect examples

$L_3(2) \pmod{2}$
Generalized coclass trees

Examples

Perfect examples

$SL_2(7)$

$SL_2(7) \mod 7$
Generalized coclass trees

Examples

Perfect examples

$SL_2(7)$

$SL_2(7) \mod 7$