Chief factors allow a group to be studied by its representation theory on particularly natural irreducible modules.
Outline

- What is a chief factor?
- How do special groups act on their chief factors?
- Structure of factor centralizers
- Examples and proof methods
Chief series

- A **chief series** of a finite group $G$ is a chain of $G$-normal subgroups $1 = H_0 < H_1 < \ldots < H_n = G$ such that there are no $G$-normal subgroups strictly contained between $H_i$ and $H_{i+1}$.

- The quotient groups $H_{i+1}/H_i$ are called **chief factors**.

- Example: $\text{Sym}(4)$ has a unique chief series

  $$1 < K_4 < A_4 < S_4.$$  

  It has chief factors $2 \times 2$, $3$, and $2$.

- Example: $\text{SL}(2,3) = C_3 \rtimes Q_8$ has a unique chief series

  $$1 < 2 < Q_8 < \text{SL}(2,3).$$

  It has chief factors $2$, $2 \times 2$, and $3$. 
Structure of factors

- A finite chief factor is a direct product of isomorphic simple groups.

- A soluble chief factor is a vector space over $\mathbb{Z}/p\mathbb{Z}$ for a prime $p$.

- In fact, a soluble chief factor $H/K$ is an irreducible $G/H$-module.

- The action of $G$ on a finite insoluble chief factor restricts to isomorphisms of one direct factor to another.

- The permutation action on the direct factors is transitive.
Size of factors

- The top factor is always simple, but the other factors can be large

- $C_p$ has an irreducible module $V$ of dimension $p - 1$ over $\mathbb{Z}/q\mathbb{Z}$, for some prime $q$

- $G = C_p \rtimes V$ has a unique chief series $1 < V < G$

- The bottom chief factor is a direct product of $p - 1$ direct factors each isomorphic to $\mathbb{Z}/q\mathbb{Z}$

- $C_p$ has a transitive action on $p$-points

- The wreath product $G = C_p \rtimes A_5^p$ has a unique chief series $1 < A_5^p < G$

- The bottom chief factor is a direct product of $p$ direct factors each isomorphic to $A_5$
A finite group is **nilpotent** iff it acts trivially on all of its chief factors.

A finite group is **supersoluble** iff its chief factors are one dimensional.

A finite group is **soluble** iff its chief factors are vector spaces.

**pd-chief factor** is a chief factor whose order is divisible by $p$.

A finite group is **$p$-nilpotent** iff it acts trivially on all of its $pd$-chief factors.

A finite group is **$p$-supersoluble** iff its $pd$-chief factors are one dimensional.

A finite group is **$p$-soluble** iff its $pd$-chief factors are vector spaces.
Centralizers of chief factors

- Define $F(G) = \bigcap\{C_G(H/K) : H/K \text{ is a chief factor}\}$

- $F(G)$ also equal to intersection of centralizers of chief factors in just one chief series

- $F(G)$ is the unique largest nilpotent $G$-normal subgroup of $G$

- Define $F_p(G) = \bigcap\{C_G(H/K) : H/K \text{ is a pd-chief factor}\}$

- $F_p(G)$ also equal to intersection of centralizers of pd-chief factors in just one chief series

- $F_p(G)$ is the unique largest $p$-nilpotent $G$-normal subgroup of $G$
Insoluble chief factors

- Every inner automorphism of a soluble chief factor is trivial

- A group is **quasi-nilpotent** if it acts as inner automorphisms on each of its chief factors

Define

$$I_G(H/K) = \{ g \in G : g \text{ acts as inner automorphism of } H/K \}$$

Define $$F^*(G) = \bigcap \{ I_G(H/K) : H/K \text{ is a chief factor} \}$$

$$F^*(G)$$ is the unique largest quasi-nilpotent $$G$$-normal subgroup of $$G$$
Structure of $F$s

- Define $O_p(G)$ the unique largest $G$-normal $p$-subgroup of $G$
- $O_p(G)$ also the unique largest $G$-subnormal $p$-subgroup of $G$
- $O_p(G)$ also the intersection of all Sylow $p$-subgroups
- $O_p(G)$ also the Sylow $p$-subgroup of $F(G)$
- $F(G) = \prod_p O_p(G)$

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- Define $O_{p'}(G)$ the unique largest $G$-normal $p'$-subgroup of $G$
- $O_{p'}(G)$ also the unique largest $G$-subnormal $p'$-subgroup of $G$
- If $G$ $p$-soluble, $O_{p'}(G)$ also the intersection of all Hall $p'$-subgroups
- $O_{p'}(G)$ also the Hall $p'$-subgroup of $F_p(G)$
- $F_p(G)/O_{p'}(G) = O_p(G/O_{p'}(G))$
Define $F^{n+1}(G)/F^n(G) = F(G/F^n(G))$ and $F^0(G) = 1$

A finite group $G$ is **soluble** iff $G = F^n(G)$ for some $n$

Define $F_p^{n+1}(G)/F_p^n(G) = F_p(G/F_p^n(G))$ and $F_p^0(G) = 1$

A finite group $G$ is **$p$-soluble** iff $G = F_p^n(G)$ for some $n$

Define $F^*_p(G)/F^*_n(G) = F^*(G/F^*_n(G))$ and $F_p^*(G) = 1$

Every finite group satisfies $G = F^*_n(G)$ for some $n$
More structure of the $F$s

- Every $p$-nilpotent group has a normal Hall $p'$-subgroup ($p$-nilpotent $= p'$-closed)

- Every $p$-nilpotent group is $P \ltimes O_{p'}(G)$

- A group is nilpotent iff it is $p$-nilpotent for all primes $p$

Let $\gamma_\infty(G) = \bigcap \gamma_n(G)$ be the intersection of the lower central series of $G$

- $E(G) = \gamma_\infty(F^*(G))$ is the unique largest perfect quasi-nilpotent $G$-subnormal subgroup of $G$

- $F^*(G)$ is a central product of $E(G)$ and $F(G)$

- $F^*(G)/F(G) = E(G/F(G))$
If $K$ is $p'$-closed normal subgroup of $G$, and $G/K$ is a $p'$-group, then call $G$ $p$-special.

A $p$-soluble group $G$ is $p$-special if and only if a Sylow $p$-subgroup acts centrally on every $pd$-chief factor.

A soluble group $G$ is $p$-special for all primes $p$:

if and only if for some Sylow system $\{P_i : i \in \pi(G)\}$, $P_i P_j' = P_j' P_i$ for all $i, j \in \pi(G)$,

if and only if for every Sylow system $\{P_i\}$ and every set of characteristic subgroups $Q_i \text{ char } P_i$, $Q_i Q_j = Q_j Q_i$ for all $i, j \in \pi(G)$. 
Examples

- Every \( \pi \)-group is \( \pi \)-closed, \( \pi' \)-closed, \( \pi' \)-nilpotent. It is \( \pi \)-nilpotent if and only if it is nilpotent.

- Given any \( \pi \)-closed group \( Q \), there is a group \( G \) with \( \pi \)-closed normal subgroup \( K \), and \( Q = G/K \), where \( G \) is not \( \pi \)-closed. The example is \( G = C_p \wr Q = (C_p^{\mid Q\mid}) \rtimes Q \) with \( p \notin \pi \).

- If \( K \) is a \( \pi \)-group, then a group \( G \) with normal subgroup \( K \) is \( \pi \)-closed if and only if the quotient is \( \pi \)-closed.

- If \( K = D_8 \) and \( p \) is an odd prime, then \( K \) is \( p \)-closed, and a group \( G \) with normal subgroup \( K \) is \( p \)-closed if and only if the quotient is \( p \)-closed.

- If \( K \) is \( \pi \)-closed but not a \( \pi \)-group, and \( |\text{Aut}(K/O_{\pi}(K))| \) is divisible by some prime in \( \pi \), then there is a group \( G \) with normal subgroup \( K \), such that \( G/K \) is \( \pi \)-closed, but \( G \) is not.
Showing a group is $\pi$-closed

- Two methods: counting Hall $\pi$-subgroups, and fusion methods

- L. Sylow (1872): The number of Sylow $p$-subgroups divides the order of the group and is equivalent to 1 mod $p$

- Every group of order $2p$ is $p$-closed, because the only divisor of 2 that is congruent to 1 mod $p$ is 1.

- P. Hall (1928): The number of Sylow $p$-subgroups in a soluble group is a product of orders of chief factors each congruent to 1 mod $p$

- If $G$ is soluble of order $3^4 \cdot 5$, then it need not be 5-closed since $1 \equiv 3^4 \mod 5$. If $G$ is supersoluble of order $3^4 \cdot 5$, then it is 5-closed, since $1 \not\equiv 3 \mod 5$.

- Vera López (1986): In a $\pi$-soluble group, the number of Hall $\pi$-subgroups is a product of orders of chief factors each congruent to 1 modulo $p$ for some $p \in \pi$
Fusion methods

- If $H \leq G$, $x, x^g \in H$, $g \in G$, then we say that $x$ and $x^g$ (properly) fuse from $H$ to $G$ if there is no $h \in H$ with $x^g = x^h$. Fused elements are conjugate in the larger group, but not in the smaller.

- In a $\pi$-closed group, there is no proper fusion from a Hall $\pi'$-subgroup to the whole group.

- (Frobenius): A group is $p'$-closed if and only if there is no proper fusion from a Sylow $p$-subgroup to the whole group.

- (Burnside 189?): If a group $G$ has an abelian Hall $p$-subgroup $Q$, then it is $p'$-closed if and only if $N_G(Q) = C_G(Q)$. Similar statements hold for Hall $\pi$-subgroups in $\pi'$-soluble groups.

- For supersoluble groups: A group is 2-nilpotent if and only if there is no proper fusion of elements of orders 2 or 4 from a Sylow 2-subgroup to the whole group.
Summary

- Chief factors are the irreducible “modules” for groups and have a simple structure.

- Many standard and interesting properties of group are equivalent to conditions of the actions on chief factors.

- The elements that act trivially form subgroups with very nice properties.

- There are a wealth of examples, and a variety of proof methods.

The End