Chief factors

Jack Schmidt

University of Kentucky

2008-03-05

Chief factors allow a group to be studied by its representation theory on particularly natural irreducible modules.

Outline

- What is a chief factor?
- How do special groups act on their chief factors?
- Structure of factor centralizers
- Examples and proof methods

Chief series

- A **chief series** of a finite group G is a chain of G-normal subgroups $1 = H_0 < H_1 < \ldots < H_n = G$ such that there are no G-normal subgroups strictly contained between H_i and H_{i+1}

- The quotient groups H_{i+1}/H_i are called **chief factors**

- Example: Sym(4) has a unique chief series

$$1 < K_4 < A_4 < S_4.$$

  It has chief factors 2×2, 3, and 2.

- Example: SL(2,3) $= C_3 \ltimes Q_8$ has a unique chief series

$$1 < 2 < Q_8 < SL(2,3).$$

  It has chief factors 2, 2×2, and 3.

Structure of factors

- A finite chief factor is a direct product of isomorphic simple groups

---

- A soluble chief factor is a vector space over $\mathbb{Z}/p\mathbb{Z}$ for a prime p
- In fact, a soluble chief factor H/K is an irreducible G/H-module

---

- The action of G on a finite insoluble chief factor restricts to isomorphisms of one direct factor to another
- The permutation action on the direct factors is transitive

Size of factors

- The top factor is always simple, but the other factors can be large

---

- C_p has an irreducible module V of dimension $p-1$ over $\mathbb{Z}/q\mathbb{Z}$, for some prime q
- $G = C_p \ltimes V$ has a unique chief series $1 < V < G$
- The bottom chief factor is a direct product of $p-1$ direct factors each isomorphic to $\mathbb{Z}/q\mathbb{Z}$

- C_p has a transitive action on p-points
- The wreath product $G = C_p \ltimes A_5^p$ has a unique chief series $1 < A_5^p < G$
- The bottom chief factor is a direct product of p direct factors each isomorphic to A_5

Action on factors

- A finite group is **nilpotent** iff it acts trivially on all of its chief factors
- A finite group is **supersoluble** iff its chief factors are one dimensional
- A finite group is **soluble** iff its chief factors are vector spaces

---

- A p**d-chief factor** is a chief factor whose order is divisible by p
- A finite group is p-**nilpotent** iff it acts trivially on all of its pd-chief factors
- A finite group is p-**supersoluble** iff its pd-chief factors are one dimensional
- A finite group is p-**soluble** iff its pd-chief factors are vector spaces

Centralizers of chief factors

- Define $F(G) = \bigcap\{C_G(H/K) : H/K \text{ is a chief factor }\}$
- $F(G)$ also equal to intersection of centralizers of chief factors in just one chief series
- $F(G)$ is the unique largest nilpotent G-normal subgroup of G

- Define $F_p(G) = \bigcap\{C_G(H/K) : H/K \text{ is a } p\text{d-chief factor }\}$
- $F_p(G)$ also equal to intersection of centralizers of pd-chief factors in just one chief series
- $F_p(G)$ is the unique largest p-nilpotent G-normal subgroup of G

Insoluble chief factors

- Every inner automorphism of a soluble chief factor is trivial
- A group is **quasi-nilpotent** if it acts as inner automorphisms on each of its chief factors

---

- Define
  $I_G(H/K) = \{g \in G : g \text{ acts as inner automorphism of } H/K\}$
- Define $F^*(G) = \bigcap\{I_G(H/K) : H/K \text{ is a chief factor }\}$
- $F^*(G)$ is the unique largest quasi-nilpotent G-normal subgroup of G

Structure of Fs

- Define $O_p(G)$ the unique largest G-normal p-subgroup of G
- $O_p(G)$ also the unique largest G-subnormal p-subgroup of G
- $O_p(G)$ also the intersection of all Sylow p-subgroups
- $O_p(G)$ also the Sylow p-subgroup of $F(G)$
- $F(G) = \prod_p O_p(G)$

- Define $O_{p'}(G)$ the unique largest G-normal p'-subgroup of G
- $O_{p'}(G)$ also the unique largest G-subnormal p'-subgroup of G
- If G p-soluble, $O_{p'}(G)$ also the intersection of all Hall p'-subgroups
- $O_{p'}(G)$ also the Hall p'-subgroup of $F_p(G)$
- $F_p(G)/O_{p'}(G) = O_p(G/O_{p'}(G))$

F series

- Define $F^{n+1}(G)/F^n(G) = F(G/F^n(G))$ and $F^0(G) = 1$
- A finite group G is **soluble** iff $G = F^n(G)$ for some n

---

- Define $F_p^{n+1}(G)/F_p^n(G) = F_p(G/F_p^n(G))$ and $F_p^0(G) = 1$
- A finite group G is p-**soluble** iff $G = F_p^n(G)$ for some n

---

- Define $F_{n+1}^*(G)/F_n^*(G) = F^*(G/F_n^*(G))$ and $F_p^*(G) = 1$
- Every finite group satisfies $G = F_n^*(G)$ for some n

More structure of the Fs

- Every p-nilpotent group has a normal Hall p'-subgroup (p-nilpotent $=$ p'-closed)
- Every p-nilpotent group is $P \ltimes O_{p'}(G)$
- A group is nilpotent iff it is p-nilpotent for all primes p

---

- Let $\gamma_\infty(G) = \bigcap \gamma_n(G)$ be the intersection of the lower central series of G
- $E(G) = \gamma_\infty(F^*(G))$ is the unique largest perfect quasi-nilpotent G-subnormal subgroup of G
- $F^*(G)$ is a central product of $E(G)$ and $F(G)$
- $F^*(G)/F(G) = E(G/F(G))$

Sylow systems and highly permutable groups

- If K is p'-closed normal subgroup of G, and G/K is a p'-group, then call G p-**special**.

- A p-soluble group G is p-special if and only if a Sylow p-subgroup acts centrally on every pd-chief factor.

- A soluble group G is p-special for all primes p:

- if and only if for some Sylow system $\{P_i : i \in \pi(G)\}$, $P_i P_j' = P_j' P_i$ for all $i, j \in \pi(G)$,

- if and only if for every Sylow system $\{P_i\}$ and every set of characteristic subgroups Q_i char P_i, $Q_i Q_j = Q_j Q_i$ for all $i, j \in \pi(G)$.

Examples

- Every π-group is π-closed, π'-closed, π'-nilpotent. It is π-nilpotent if and only if it is nilpotent.

- Given any π-closed group Q, there is a group G with π-closed normal subgroup K, and $Q = G/K$, where G is not π-closed. The example is $G = C_p \wr Q = (C_p^{|Q|}) \rtimes Q$ with $p \notin \pi$.

- If K is a π-group, then a group G with normal subgroup K is π-closed if and only if the quotient is π-closed.

- If $K = D_8$ and p is an odd prime, then K is p-closed, and a group G with normal subgroup K is p-closed if and only if the quotient is p-closed.

- If K is π-closed but not a π-group, and $|\operatorname{Aut}(K/O_\pi(K))|$ is divisible by some prime in π, then there is a group G with normal subgroup K, such that G/K is π-closed, but G is not.

Showing a group is π-closed

- Two methods: counting Hall π-subgroups, and fusion methods

- L. Sylow (1872): The number of Sylow p-subgroups divides the order of the group and is equivalent to 1 mod p
- Every group of order $2p$ is p-closed, because the only divisor of 2 that is congruent to 1 mod p is 1.

- P. Hall (1928): The number of Sylow p-subgroups in a soluble group is a product of orders of chief factors each congruent to 1 mod p
- If G is soluble of order $3^4 \cdot 5$, then it need not be 5-closed since $1 \equiv 3^4 \mod 5$. If G is supersoluble of order $3^4 \cdot 5$, then it is 5-closed, since $1 \not\equiv 3 \mod 5$.

- Vera López (1986): In a π-soluble group, the number of Hall π-subgroups is a product of orders of chief factors each congruent to 1 modulo p for some $p \in \pi$

Fusion methods

- If $H \leq G$, $x, x^g \in H$, $g \in G$, then we say that x and x^g (properly) **fuse** from H to G if there is no $h \in H$ with $x^g = x^h$. Fused elements are conjugate in the larger group, but not in the smaller.
- In a π-closed group, there is no proper fusion from a Hall π'-subgroup to the whole group.
- (Frobenius): A group is p'-closed if and only if there is no proper fusion from a Sylow p-subgroup to the whole group.
- (Burnside 189?): If a group G has an abelian Hall p-subgroup Q, then it is p'-closed if and only if $N_G(Q) = C_G(Q)$. Similar statements hold for Hall π-subgroups in π'-soluble groups.
- For supersoluble groups: A group is 2-nilpotent if and only if there is no proper fusion of elements of orders 2 or 4 from a Sylow 2-subgroup to the whole group.

Summary

- Chief factors are the irreducible “modules” for groups and have a simple structure
- Many standard and interesting properties of group are equivalent to conditions of the actions on chief factors
- The elements that act trivially form subgroups with very nice properties
- There are a wealth of examples, and a variety of proof methods

THE END