Finite groups with a unique subgroup of order p

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Abstract

This is a toy example to illustrate some of the techniques of fusion and transfer as described in Gorenstein's text Finite Groups. Finite groups with a unique subgroup of order p (p an odd prime) are classified in terms of a cyclic p'-extension of relatively simple combinatorial data.

This note is intended to address a slightly misstated exercise in somewhat more detail and to be a toy example of the theory of fusion. Likely this classification was still child's play in the 1930s, but it does give a clean, explicit description of a reasonably natural sounding class. The original misstated exercise was to show that if a finite group had a unique subgroup of order p, for some prime p, then that subgroup was central. As the non-abelian group of order 6 and $p = 3$ shows, this is an absurd claim. However, the groups that do occur break down naturally into the central case and a cyclic extension. The central case is elegantly described by Burnside's N/C theorem. Notation and concepts are as in [**1**], especially chapters 5 and 7.

Definition 1. *A finite group is called* p***-nilpotent*** *if it has a normal subgroup of order coprime to* p *and index a power of* p *(*p *some prime). A finite group that is not* p*-nilpotent is called* p***-length 1*** *if it has a normal* p*-nilpotent subgroup of index coprime to* p*.*

We will "p-nilpotent or p-length 1" to just "p-length at most 1". The name p-nilpotent comes from several important similarities of p-nilpotent groups to nilpotent groups, the easiest to describe is simply that a finite group is nilpotent if and only if it is p-nilpotent for all primes p. Another way to describe p-nilpotent groups is as those groups which have a normal p-complement. These groups have been studied for more than 100 years, and are involved in one of the earliest results in "fusion":

Lemma 2 (Burnside). *If a Sylow* p*-subgroup is centralized by its normalizer, then the whole group is* p*-nilpotent.*

Proof. This is [**1**, Th. 7.4.3, p.252]. □

In fact modern methods (as in 1930s) have improved Burnside's result to:

Lemma 3. *If a Sylow* p*-subgroup* P *of* G *is abelian and* $N = N_G(P)$*, then* $P = (P \cap N') \times (P \cap Z(N))$*.*

Proof. This is [**1**, Th. 7.4.4, p.253]. □

We also need the well-known classification in case G is itself a p-group:

2000 Mathematics Subject Classification 20D20 (20D10)

LEMMA 4. *If G is a p-group with a unique subgroup of order p for an odd prime p, then G is cyclic.*

Proof. This is [1, Th. 5.4.10.ii, p.199]. □

We can put these together to get:

PROPOSITION 5. *If G has a unique subgroup of order p for an odd prime p, then either G is p-nilpotent, or $O_p(G) \cap Z(G) = 1$. In particular, $C_G(\Omega(O_p(G)))$ is p-nilpotent.*

Proof. By lemma 4, the Sylow p-subgroup P of G is cyclic, since it too has only one subgroup of order p. By lemma 3, P decomposes as a direct product of $P \cap N'$ and $P \cap Z(N)$. Since a cyclic p-group is directly indecomposable, either $P \leqslant Z(N)$ or $P \cap Z(N) = 1$. In the former case, lemma 2 shows that G is p-nilpotent. In the latter case one has that $O_p(G) \cap Z(G) = P \cap Z(G) \leqslant P \cap Z(N) = 1$. The final statement follows since $\Omega(O_p(G))$, the unique subgroup of order p, is central, so one is in the first case. □

We can turn this into a structure theorem in the p-nilpotent case:

PROPOSITION 6. *For each odd prime p, there is a 1-1 correspondence between isomorphism classes of finite p-nilpotent groups with a unique subgroup of order p and triples (Q, α, n) where Q is a representative of an isomorphism class of finite groups of odd order, α is a conjugacy class representative of a class of p-elements in $\mathrm{Aut}(Q)$, and n is a positive integer.*

Proof. Given a p-nilpotent group G with a unique subgroup $\Omega(O_p(G))$ of order p, let $Q = O_{p'}(G)$ be the largest normal subgroup of order coprime to p, and let P be a Sylow p-subgroup and complement to Q. By lemma 4 P is cyclic, say $P = \langle x \rangle$. Hence there is some automorphism $\alpha \in \mathrm{Aut}(Q)$ of order p^k for some nonnegative integer k such that $g^x = g^\alpha$ for all $g \in Q$. Since $[Q, \Omega(O_p(G))] \leqslant Q \cap \Omega(O_p(G)) = 1$, the order of x must be strictly larger than the order of α as the powers of x of order p centralize Q. Let n be the positive integer defined by the order of x being equal to p^{k+n}. This constitutes the map from isomorphism classes of G to triples (Q, α, n). Clearly it is well-defined, as the choices commute with isomorphisms.

Given a triple (Q, α, n) with α of order p^k, define G to be the semi-direct product a Q with a cyclic group $P = \langle x \rangle$ of order p^{k+n} acting on Q by $g^x = g^\alpha$ for all $g \in Q$. This constitutes the map from triples to isomorphism classes of groups. Again, the choices of conjugacy result in isomorphic groups. It is also clear that such a group G is p-nilpotent, and since the Sylow p-subgroup P is cyclic and since $\Omega(P)$ is normal in G, it is clearly the unique subgroup of order p in G. Hence the map is well-defined.

It remains to check that the two maps are inverses of each other, but this is clear. □

This is sufficient to prove the structure theorem:

THEOREM 7. *Every finite group with a unique subgroup of order p is a extension of a normal p-nilpotent subgroup with a unique subgroup of order p by a cyclic quotient of order dividing $p - 1$ and acting faithfully, and conversely every such extension has a unique subgroup of order p. In particular, every such group is of p-length at most 1.*

Proof. This is just because $C_G(\Omega(O_p(G)))$ is a normal, p-nilpotent subgroup with a unique subgroup of order p, and $G/C_G(\Omega(O_p(G)))$ is isomorphic to a subgroup of the automorphism group of a group of order p, which is cyclic of order $p-1$. Conversely, the extension still normalizes $\Omega(P)$ for the cyclic Sylow p-subgroup P. The final statement is clear. □

It is important to notice that such groups need not be p-nilpotent, and they may or may not have non-identity normal subgroups of order coprime to p, so that there are no obvious simplifications to the structure theorem.

References

1. Daniel Gorenstein, *Finite groups* (Chelsea Publishing Co., New York, 1980), 2nd edn. ISBN 0-8284-0301-5.

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