Name: ______________________________

MA162-020
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Quiz 4.1: Reading Tableaux for the Simplex Algorithm

Write the following table as a system of equations. Write the solution for P in terms of free variables. Does setting the free variables to 0 give you the maximal possible P? Why or why not? Here $X, Y, U, V \geq 0$ are constrained to be non-negative.

X	Y	U	P	RHS
1	2	3	0	10
0	4	5	1	11

Basic variables:

P =

Free variables:

Maximum?

X	Y	U	V	P	RHS
2	1	3	0	0	10
4	0	5	1	0	11
6	0	7	0	1	12

Basic variables:

P =

Free variables:

Maximum?

X	Y	U	V	P	RHS
2	1	3	0	0	10
4	0	5	1	0	11
-6	0	7	0	1	12

Basic variables:

P =

Free variables:

Maximum?

Examples 4.1: Pivoting makes business decisions easier (I)

Ivy Dahoe oversees two of the family mines in the northern United States and tries to minimize costs while meeting the family businesses production goals. The Alene mine costs \$15,000 per day to operate and produces 3000 oz. of silver, 25 tons of lead, and 30 tons of zinc. The Blackfoot mine costs \$18,000 per day to operate and produces 1800 oz. of silver, 30 tons of lead, and 27 tons of zinc. This quarter the family business production demands 60,000 oz. of silver, 650 tons of lead, and 650 tons of zinc. Ivy wants to hire you to help run the business, but to prove you can handle it, she wants to know how many days you think she should run each mine in order to meet the quarterly production goals while minimizing the cost.

Decision variables:
(that we are **free** to decide)
A = # of days to run Alene
B = # of days to run Blackfoot

Slack variables:
(our basic variables that are **determined**)
S = thousands of ounces of surplus Silver
L = tons of surplus Lead
Z = tons of surplus Zinc

Objective:
C = thousands of dollars of cost

Constraints: $A, B, S, L, Z \geq 0$

Equations:

$$\begin{array}{rrrrrrl}
-3A & -\ 1.8B & +\ S & & & = & -60 \\
-25A & -\ 30B & & +\ L & & = & -650 \\
-30A & -\ 27B & & & +\ Z & = & -650 \\
\hline
-15A & -\ 18B & & & & +\ C = & 0
\end{array}$$

A	B	S	L	Z	C	RHS
-3	-1.8	1	0	0	0	-60
-25	-30	0	1	0	0	-650
-30	-27	0	0	1	0	-650
-15	-18	0	0	0	1	0

Solution:

$$\begin{array}{ll}
S = \ 3A + 1.8B - 60 & C = 15A + 18B \\
L = 25A + 30B - 650 & A = \text{free} \\
Z = 30A + 27B - 650 & B = \text{free}
\end{array}$$

In the initial setup of this problem the business decision we need to make is hard: we need to decide how many days to run the mines. We have to do that, because A, B are the **free** variables in our setup. It would be nice if one of the other variables was free. Instead of having A be free and L be determined, let's just **swap their roles**! We'll make A be a pivot, and let L be free.

$$\begin{array}{rr|rrrr|r}
A & B & S & L & Z & C & \text{RHS} \\
\hline
-3 & -1.8 & 1 & 0 & 0 & 0 & -60 \\
-25 & -30 & 0 & 1 & 0 & 0 & -650 \\
-30 & -27 & 0 & 0 & 1 & 0 & -650 \\
\hline
-15 & -18 & 0 & 0 & 0 & 1 & 0
\end{array}
\xrightarrow[\substack{R_3 + 30R_2 \\ R_4 + 15R_2}]{\substack{-R_2/25 \\ R_1 + 3R_2}}
\begin{array}{rr|rrrr|r}
A & B & S & L & Z & C & \text{RHS} \\
\hline
0 & 1.8 & 1 & -0.12 & 0 & 0 & 18 \\
1 & 1.2 & 0 & -0.04 & 0 & 0 & 26 \\
0 & 9.0 & 0 & -1.20 & 1 & 0 & 130 \\
\hline
0 & 0.0 & 0 & -0.60 & 0 & 1 & 390
\end{array}
\xrightarrow{\text{solve}}$$

$$\left\{\begin{array}{ll}
A = 26 - 1.2B + 0.04L & L = \text{free} \\
B = \text{free} & Z = 130 - \ 9B + 1.2L \\
S \ = 18 - 1.8B + 0.12L & C = 390 + 0.6L
\end{array}\right.
\xrightarrow{B=0, L=0 \text{ is easy}}
\left\{\begin{array}{ll}
A = 26 & L = 0 \\
B = 0 & Z = 130 \\
S \ = 18 & C = 390
\end{array}\right.$$

The bigger L is, the higher the cost, and $L = 0$ already meets our production goals so $L = 0$ is clearly the right choice! We have more freedom in B. We could choose $B = 0$, and get a lot of surplus, or we could choose $B = 10$, to get $A = 14$, $B = 10$, $S = 0$, $L = 0$, $Z = 40$, and so have less surplus. See how we have switched to an easier business decision? "Big L = big cost, L = 0 good" and "B does not affect cost, B = 0 fine".

Examples 4.1: Pivoting makes business decisions easier (II)

Mr. Marjoram runs a stuffed animal factory, and is very worried about making some money using his rather large inventory of plush fabric, cloud-like stuffing, and whimsical trim. He decides he is going to use his inventory wisely to make the 2010 Marjoram Menagerie as profitable as possible! His menagerie only includes Pandas, Saint Bernards, and Onery Ostriches. Each Panda requires 1.5 square yards of plush, 30 cubic feet of stuffing, and 12 pieces of trim. Each Saint Bernard requires 2 square yards of plush, 35 cubic feet of stuffing, and 8 pieces of trim. Each Onery Ostrich requires 2.5 square yards of plush, 25 cubic feet of stuffing, and 5 pieces of trim. Marjoram's storage room has 110 square yards of velvety plush, 1400 cubic feet of fluffy stuffing, and 350 pieces of tremendous trim. He expects to make \$10 profit for each Panda produced, \$15 profit for each Saint Bernard, and \$12 profit for each Onery Ostrich produced. How many stuffed animals of each type should he make in order to maximize his (expected) profit?

Decision variables:
(that we are **free** to decide)
X = # of Pandas to produce
Y = # of Saint Bernards to produce
Z = # of Onery Ostriches to produce

Slack variables:
(our basic variables that are **determined**)
F = square yards of surplus plush Fabric
S = cubic feet of surplus Stuffing
T = # of pieces of surplus Trim

Objective:
P = dollars of profit

Constraints: $X, Y, Z, F, S, T \geq 0$

Equations:

$$\begin{array}{rrrrrrrl}
1.5X + & 2Y + & 2.5Z + & F & & & & = 110 \\
30X + & 35Y + & 25Z & & + S & & & = 1400 \\
12X + & 8Y + & 5Z & & & + T & & = 350 \\
\hline
-10X - & 15Y - & 12Z & & & & + P & = 0
\end{array}$$

X	Y	Z	F	S	T	P	RHS
1.5	2	2.5	1	0	0	0	110
30	35	25	0	1	0	0	1400
12	8	5	0	0	1	0	350
−10	−15	−12	0	0	0	1	0

Solution:

$F = 110 - 1.5X - 2Y - 2.5Z$
$S = 1400 - 30X - 35Y - 25Z$
$T = 350 - 12X - 8Y - 5Z$
$P = 10X + 15Y + 12Z$

X = free
Y = free
Z = free

The current situation leaves us with a difficult business decision: how many stuffed animals should we make? We are forced to guess values for X, Y, and Z, because they are our free variables. We want to pivot, to swap the other variables to be free (and hopefully, freely set them to 0!).

X	Y	Z	F	S	T	P	RHS
1.5	2	2.5	1	0	0	0	110
30	35	25	0	1	0	0	1400
12	8	5	0	0	1	0	350
−10	−15	−12	0	0	0	1	0

$\xrightarrow[R_3 - 5R_1,\ R_4 + 12R_1]{R_1/2.5,\ R_2 - 25R_1}$

X	Y	Z	F	S	T	P	RHS
0.6	0.8	1	0.4	0	0	0	44
15	15	0	−10.0	1	0	0	300
9	4	0	−2.0	0	1	0	130
−2.8	−5.4	0	4.8	0	0	1	528

$\xrightarrow[R_3 - 4R_2,\ R_4 + 5.4R_2]{R_2/15,\ R_1 - 0.8R_2}$

X	Y	Z	F	S	T	P	RHS
−0.2	0	1	0.93	−0.05	0	0	28
1.0	1	0	−0.67	0.07	0	0	20
5.0	0	0	0.67	−0.27	1	0	50
2.6	0	0	1.20	0.36	0	1	636

$$\xrightarrow{\text{solve}} \begin{cases} Z &= 28 + 0.2X - 0.93F + 0.05S \\ Y &= 20 - X + 0.67F - 0.07S \\ T &= 50 - 5X - 0.67F + 0.27S \\ P &= 636 - 2.6X - 1.2F - 0.36S \end{cases}$$

X = free
F = free
S = free

Since $X, F, S \geq 0$, but they only subtract from the profit P, the business decision is very easy: $X = F = S = 0$ is the only smart choice! This leaves $Z = 28$, $Y = 20$, $T = 50$, and $P = 636$.

Examples 4.1: Pivoting makes business decisions easier (III)

Boise Lumber has decided to enter the lucrative pre-fabricated housing business. Initially it plans to offer three models: standard, deluxe, and luxury. Each house is prefabricated and partially assembled in the factory, and the final assembly is completed on site. The standard model costs \$6000 in materials, requires 240 hours of factory labor, 180 hours of on-site labor, and sells for a \$3400 profit. The deluxe model requires \$8000 in materials, 220 hours of factory labor, 210 hours of on-site labor, and sells for a \$4000 profit. The luxury model requires \$10000 in materials, 200 hours of factory labor, and 300 hours of on-site labor. Boise Lumber has allocated \$8.2 million in materials, 218 thousand factory hours, and 237 thousand on-site hours for the first year of production. How many houses of each type should be produced in order to maximize the profit while staying within the budget?

Decision variables:
(that we are **free** to decide)
X = # of Standard models to produce
Y = # of Deluxe models to produce
Z = # of Luxury models to produce

Slack variables:
(our basic variables that are **determined**)
M = thousands of dollars of surplus materials
F = hours of surplus factory labor
S = hours of surplus on-site labor

Objective:
P = thousands of dollars of profit

Constraints: $X, Y, Z, M, F, S \geq 0$

Equations:

$$\begin{array}{rrrrrrrcr}
6X & + & 8Y & + & 10Z & + M & & = & 8200 \\
240X & + & 220Y & + & 200Z & & + F & = & 218000 \\
180X & + & 210Y & + & 300Z & & + S & = & 237000 \\
\hline
-3.4X & - & 4Y & - & 5Z & & + P & = & 0
\end{array}$$

$$\begin{array}{rrr|rrrr|r}
X & Y & Z & M & F & S & P & \text{RHS} \\
\hline
6 & 8 & 10 & 1 & 0 & 0 & 0 & 8200 \\
240 & 220 & 200 & 0 & 1 & 0 & 0 & 218000 \\
180 & 210 & 300 & 0 & 0 & 1 & 0 & 237000 \\
\hline
-3.4 & -4 & -5 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Solution:

$M = 8200 - 6X - 8Y - 10Z$
$F = 218000 - 240X - 220Y - 200Z$
$S = 237000 - 180X - 210Y - 300Z$
$P = 3.4X + 4Y + 5Z$

$X =$ free
$Y =$ free
$Z =$ free

Again the "solution" still leaves us with the difficult problem of deciding how many houses of each model should be produced.

$$\begin{array}{rrr|rrrr|r}
X & Y & Z & M & F & S & P & \text{RHS} \\
\hline
6 & 8 & 10 & 1 & 0 & 0 & 0 & 8200 \\
240 & 220 & 200 & 0 & 1 & 0 & 0 & 218000 \\
180 & 210 & 300 & 0 & 0 & 1 & 0 & 237000 \\
\hline
-3.4 & -4 & -5 & 0 & 0 & 0 & 1 & 0
\end{array}
\xrightarrow[\substack{R_2 - 200R_3 \\ R_4 + 5R_3}]{\substack{R_3/300 \\ R_1 - 10R_3}}
\begin{array}{rrr|rrrr|r}
X & Y & Z & M & F & S & P & \text{RHS} \\
\hline
0 & 1 & 0 & 1 & 0 & -0.0333 & 0 & 300 \\
120 & 80 & 0 & 0 & 1 & -0.6667 & 0 & 60000 \\
0.6 & 0.7 & 1 & 0 & 0 & 0.0033 & 0 & 790 \\
\hline
-0.4 & -0.5 & 0 & 0 & 0 & 0.0167 & 1 & 3950
\end{array}
\xrightarrow[\substack{R_3 - 0.7R_1 \\ R_4 + 0.5R_1}]{R_2 - 80R_1}$$

$$\begin{array}{rrr|rrrr|r}
X & Y & Z & M & F & S & P & \text{RHS} \\
\hline
0 & 1 & 0 & 1 & 0 & -0.0333 & 0 & 300 \\
120 & 0 & 0 & -80 & 1 & 2.0000 & 0 & 36000 \\
0.6 & 0 & 1 & -0.7 & 0 & 0.0267 & 0 & 580 \\
\hline
-0.4 & 0 & 0 & 0.5 & 0 & 0.0000 & 1 & 4100
\end{array}
\xrightarrow[\substack{R_3 - 0.6R_2 \\ R_4 + 0.4R_2}]{R_2/120}
\begin{array}{rrr|rrrr|r}
X & Y & Z & M & F & S & P & \text{RHS} \\
\hline
0 & 1 & 0 & 1.0000 & 0.0000 & -0.3333 & 0 & 300 \\
1 & 0 & 0 & -0.6667 & 0.0083 & 0.0167 & 0 & 300 \\
0 & 0 & 1 & -0.3000 & -0.0050 & 0.0167 & 0 & 400 \\
\hline
0 & 0 & 0 & 0.2333 & 0.0033 & 0.0067 & 1 & 4220
\end{array}$$

After 3 pivots (role swaps), we have an easy business decision:

$$P = 4220 - 0.2333M - 0.0033F - 0.0067S$$

with $M, F, S \geq 0$. How big should we make the surpluses M, F, S? Well, the surplus directly lowers the profit, so we should make each one 0 of course!

$$\begin{cases} Y = 300 \\ X = 300 \\ Z = 400 \\ P = 4220 \end{cases}
\qquad
\begin{array}{ll}
M & = 0\ (free) \\
F & = 0\ (free) \\
S & = 0\ (free)
\end{array}$$