MA162: Finite mathematics

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January 26, 2011

Schedule:

- HW A2 is due Monday, Jan 31st, 2011.
- HW A3 is due Sunday, Feb 6th, 2011.
- Exam 1 is Monday, Feb 7th, 5:00pm-7:00pm.
  Old exams on class webpage
- HW B1 is due Monday, Feb 21st, 2011.

Today we will cover 2.2, augmented matrices, and the elimination algorithm

2.2: Efficiently solving systems

- We solved systems last time with two variables
- Real decisions involve balancing half a dozen variables
- Two main changes to handle this:
- Write down less so that we can see the important parts clearly
- Use a **systematic** method to solve

2.2: Efficient notation

- We worked some equations with the variables x, y
- We could have used M and T
- The letters we used did not matter; just placeholders
- Why do we even write them down?
- The plus signs and equals are pretty boring too.
- The only part we need are the numbers
  (and where the numbers are)

2.2: Augmented matrices

$$\begin{aligned} x + 2y + 3z &= 4 \\ y + 5z &= 7 \\ 8x + y &= 9 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 7 \\ 8 & 1 & 0 & 9 \end{array}\right]$$

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2.2: More examples

$$\begin{aligned} 2x + 3z &= 4 \\ 6z + 5y &= 7 \\ 8x + 9y &= 1 \end{aligned} \qquad \begin{aligned} 2x &+ 0y &+ 3z &= 4 \\ 0x &+ 5y &+ 6z &= 7 \\ 8x &+ 9y &+ 0z &= 1 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 0 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 8 & 9 & 0 & 1 \end{array}\right]$$

$$\begin{aligned} 4x + 3z &= 2 \\ 8z - y &= 7 \\ 5x - 9y &= 6 \end{aligned} \qquad \begin{aligned} 4x &+ 0y &+ 3z &= 2 \\ 0x &- 1y &+ 8z &= 7 \\ 5x &- 9y &+ 0z &= 6 \end{aligned} \qquad \left[\begin{array}{ccc|c} 4 & 0 & 3 & 2 \\ 0 & -1 & 8 & 7 \\ 5 & -9 & 0 & 6 \end{array}\right]$$

$$\begin{aligned} y &= 3 - 2x \\ z &= 7 + 4y \\ x &= 6 + 5z \end{aligned} \qquad \begin{aligned} 2x &+ 1y &+ 0z &= 3 \\ 0x &- 4y &+ 1z &= 7 \\ x &+ 0y &- 5z &= 6 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 1 & 0 & 3 \\ 0 & -4 & 1 & 7 \\ 1 & 0 & -5 & 6 \end{array}\right]$$

2.2: Efficient notation

- We now have a very clean way to write down systems of equations
- Make sure you can convert from a system of equations to the **augmented matrix**
- Make sure you can convert from an augmented matrix to a system of equations

2.2: A systematic procedure

- Now we will learn a method of solving systems
- We will transform the equations until they look like (REF):

$$\begin{aligned} x + 2y + 3z &= 4 \\ 5y + 6z &= 7 \\ 8z &= 9 \end{aligned}$$

- Next time, we will transform them until they look like (RREF):

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

- We will do this by following a set of rules
- Your work on the exam is graded **strictly**

2.2: First step: Find pivots

- The 0th step is to make sure you have got an augmented matrix
- Once you do we look for **pivots**
- Each row should have a pivot;
  it is the **first nonzero** number in the row

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 6 & 7 \end{array}\right]$$

- We want **one pivot per column**
- We are usually **disappointed**

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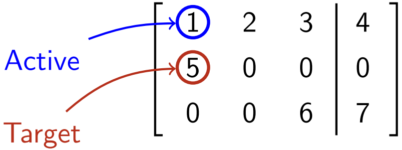
2.2: Second step: Choose target

- If there are two pivots in one column, we **eliminate** one of them
- The **active pivot** is the first pivot in the first bad column
- The **target pivot** is the next pivot in the first bad column



2.2: Third step: Eliminate the target

- We are now going to subtract a multiple of the active row from the target row
- We choose the multiple: $\frac{\text{target pivot}}{\text{active pivot}}$
- In our example, we choose $\frac{5}{1} = 5$

$$\begin{array}{rrrr} 5 & 0 & 0 & 0 \\ -5 \cdot (\ 1 & 2 & 3 & 4) \\ \hline \end{array} \qquad \begin{array}{rrrrr} & 5 & 0 & 0 & 0 \\ + & -5 & -10 & -15 & -20 \\ \hline & 0 & -10 & -15 & -20 \end{array}$$

- We changed the old 5 to a zero!
- This new row will replace our old target row

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2.2: Fourth step: regroup

- Now we rewrite our new matrix and start over with an **easier** system

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 6 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -10 & -15 & -20 \\ 0 & 0 & 6 & 7 \end{array}\right]$$

- We also need to **show our work** in a **very specific way**

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2.2: First step again: find pivots

- Now we begin again with our new **simpler** system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -10 & -15 & -20 \\ 0 & 0 & 6 & 7 \end{array}\right]$$

- We find the pivots
- Each column left of the bar has exactly one pivot!
- This is called **REF** and means that for today we are done
- We can solve this using algebra, first for z, then for y, then for x

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2.2: Final step: Back substitution

- To finish up, we convert back to a system of equations:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -10 & -15 & -20 \\ 0 & 0 & 6 & 7 \end{array}\right] \qquad \begin{array}{rrrl} x & +2y & +3z & = 4 \\ & -10y & -15z & = -20 \\ & & 6z & = 7 \end{array}$$

- We can solve for z very easily: $6z = 7$ means $z = \frac{7}{6}$

2.2: Final step: Back substitution

- We know $z = \frac{7}{6}$ and

$$\begin{aligned} x \quad + 2y \quad + 3z &= 4 \\ -10y \quad - 15z &= -20 \end{aligned}$$

- We can make the second equation easier by plugging in z:

$$-20 = -10y - 15z = -10y - 15\tfrac{7}{6} = -10y - 17.5$$

$$10y = 2.5 \qquad y = 0.25$$

- We can make the first equation easier by plugging in both y and z:

$$4 = x + 2y + 3z = x + 2 \cdot 0.5 + 3 \cdot \frac{7}{6} = x + 0.5 + 3.5 \qquad x = 0$$

- Our answer is $(x = 0, y = 0.25, z = 7/6)$

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2.2: Real question

- You have three types of workers: packers, cutters, sewers.
- You have three types of products: short-sleeve, sleeveless, long-sleeve.
- It takes the following amount of time to make them:

	Short	**Less**	**Long**
Pack	4	3	4
Cut	12	9	15
Sew	24	22	28

- You have 24 hours of packers, 80 hours of cutters, and 160 hours of sewers
- How many of each should you make to keep everyone working?

2.2: As system, as matrix

- As a system of equations:
  Make x short-sleeve, y sleeveless, z long-sleeve

$$\left\{ \begin{array}{rcrcrcr} 4x & + & 3y & + & 4z & = & 1440 \\ 12x & + & 9y & + & 15z & = & 4800 \\ 24x & + & 22y & + & 28z & = & 9600 \end{array} \right.$$

- As a matrix:

$$\left(\begin{array}{rrr|r} 4 & 3 & 4 & 1440 \\ 12 & 9 & 15 & 4800 \\ 24 & 22 & 28 & 9600 \end{array} \right)$$

2.2: REF it

$$\left(\begin{array}{ccc|c} 4 & 3 & 4 & 1440 \\ 12 & 9 & 15 & 4800 \\ 24 & 22 & 28 & 9600 \end{array}\right) \xrightarrow[R_3-6R_1]{R_2-3R_1} \left(\begin{array}{ccc|c} 4 & 3 & 4 & 1440 \\ 0 & 0 & 3 & 480 \\ 0 & 4 & 4 & 960 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 4 & 3 & 4 & 1440 \\ 0 & 4 & 4 & 960 \\ 0 & 0 & 3 & 480 \end{array}\right) \qquad \text{REF}$$

- As equations: $\left\{\begin{array}{rcrcrcr} 4x & + & 3y & + & 4z & = & 1440 \\ & + & 4y & + & 4z & = & 960 \\ & & & & 3z & = & 480 \end{array}\right.$
- $z = 480/3 = 160$, then $4y + 4(160) = 960$ and $y = 80$, then ... and $x = 140$
- So make 140 short-sleeve, 80 sleeveless, and 160 long-sleeves