

MA162: Finite mathematics

Jack Schmidt

University of Kentucky

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SCHEDULE:

- HW B2 is due Today, Feb 28th, 2011.
- HW B3 is due Sunday, Mar 6th, 2011.
- Exam 2 is Monday, Mar 7th, 5:00pm-7:00pm.

Today we will cover 4.1: Simplex algorithm

Exam 2: Overview

- 35% Ch 2, Matrix arithmetic (HWB1):
 - Sizes, Addition, subtraction, multiplication by a number, matrix-matrix multiplication, matrix to a power
 - Matrix inverse (2×2 and 3×3); Solving system given inverse
- 30% Ch 3, Two-variable linear programs (HWB2):
 - Setting up a linear programming problem
 - Solving using the graphical method
- 35% Ch 4, Simplex algorithm (HWB3):
 - **Solving primal problems using simplex algorithm**
 - Solving dual problems using simplex algorithm

4.1: Linear programming problems

- An LPP has three parts:
 - The variables (the business decision to be made)
 - The inequalities (the laws, constraints, rules, and regulations)
 - The objective (maximize profit, minimize cost)
- If there are more than two variables, graphing gets hard!
- We need to use matrices to handle so many variables.

4.1: Today's LPP as word problem

- Old MacDonald had 100 acres, \$6000, and 2400 labor hours
- Crop A costs him \$50/acre and 20hrs/acre in labor
- Crop B costs him \$60/acre and 25hrs/acre in labor
- Crop A earns him \$150/acre and Crop B earns him \$200/acre
- How many acres of each crop should he plant?

4.1: Today's LPP in summary form

- Variables:

X = Number of acres of crop A to plant

Y = Number of acres of crop B to plant

- Constraints:

$$\begin{array}{rcl} X + Y & \leq & 100 \quad \text{Land} \\ 50X + 60Y & \leq & 6000 \quad \text{Capital} \\ 20X + 25Y & \leq & 2400 \quad \text{Labor} \end{array}$$

- Objective:

$$\text{Maximize } P = 150X + 200Y$$

4.1: Inequalities are just equalities in disguise

- All of our variables are non-negative (realistic)
- Think about $X + Y \leq 100$
- It means $X + Y$ has not yet exceeded 100
- We could still add something to bump it up to 100
- $X + Y + U = 100$ with $X, Y, U \geq 0$
- U is “the rest” of the 100, the “unused” land, the **slack**

$$U = 100 - (X + Y)$$

- “ $U = 20$ ” means we left 20 acres fallow (unused)

4.1: LPP are just systems of equations in disguise

$$\begin{array}{rcl} X + Y & \leq & 100 \quad \text{Land} \\ 50X + 60Y & \leq & 6000 \quad \text{Capital} \\ 20X + 25Y & \leq & 2400 \quad \text{Labor} \end{array}$$

- Define our slack variables:

$$\begin{array}{rcl} U = 100 - (X + Y) & \text{unused Land} \\ V = 6000 - (50X + 60Y) & \text{unused Capital} \\ W = 2400 - (20X + 25Y) & \text{unused Labor} \end{array}$$

- Even profit is an equation: $P = 150X + 200Y$ just means $-150X - 200Y + P = 0$
- Now we have a system of equations:

$$\begin{array}{rcl} X + Y + U & = & 100 \quad \text{Land} \\ 50X + 60Y + V & = & 6000 \quad \text{Capital} \\ 20X + 25Y + W & = & 2400 \quad \text{Labor} \\ \hline -150X - 200Y + P & = & 0 \quad \text{Profit} \end{array}$$

4.1: Write it as a matrix

- The system of equations:

$$\begin{array}{rcl} X + Y + U = 100 & \text{Land} \\ 50X + 60Y + V = 6000 & \text{Capital} \\ 20X + 25Y + W = 2400 & \text{Labor} \\ \hline -150X - 200Y + P = 0 & \text{Profit} \end{array}$$

- Now as a matrix:

$$\begin{array}{cccccc} X & Y & U & V & W & P & RHS & \\ \left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 100 \\ 50 & 60 & 0 & 1 & 0 & 0 & 6000 \\ 20 & 25 & 0 & 0 & 1 & 0 & 2400 \\ \hline -150 & -200 & 0 & 0 & 0 & 1 & 0 \end{array} \right) & \begin{array}{l} \text{Land} \\ \text{Capital} \\ \text{Labor} \\ \text{Profit} \end{array} \end{array}$$

4.1: Analyze the matrix

- Our matrix is basically in RREF!

$$\begin{array}{cccccc|c} X & Y & U & V & W & P & RHS & \\ \hline 1 & 1 & \textcircled{1} & 0 & 0 & 0 & 100 & \text{Land} \\ 50 & 60 & 0 & \textcircled{1} & 0 & 0 & 6000 & \text{Capital} \\ 20 & 25 & 0 & 0 & \textcircled{1} & 0 & 2400 & \text{Labor} \\ \hline -150 & -200 & 0 & 0 & 0 & \textcircled{1} & 0 & \text{Profit} \end{array}$$

- U,V,W,P have pivots, and X,Y are free.
We can make X and Y whatever we want!
- Wait. What do we want them to be?
Isn't that why we started doing this anyways?
- OMG We have the wrong pivots!

4.1: Operation Row Op to the rescue!

- Well, what if we decided the 25 should have been a pivot?
- Easy to fix using row ops:

$$\begin{array}{l} R_1 - 0.04R_3 \\ R_2 - 2.4R_3 \\ R_4 + 8R_3 \\ R_3/25 \end{array} \rightarrow \left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 100 \\ 50 & 60 & 0 & 1 & 0 & 0 & 6000 \\ 20 & 25 & 0 & 0 & 1 & 0 & 2400 \\ \hline -150 & -200 & 0 & 0 & 0 & 1 & 0 \end{array} \right)$$
$$\left(\begin{array}{cccccc|c} 1/5 & 0 & \textcircled{1} & 0 & -1/25 & 0 & 4 \\ 2 & 0 & 0 & \textcircled{1} & -12/5 & 0 & 240 \\ 4/5 & \textcircled{1} & 0 & 0 & 1/25 & 0 & 96 \\ \hline 10 & 0 & 0 & 0 & 8 & \textcircled{1} & 19200 \end{array} \right)$$

- Y, U, V, P are pivots, X and W are free.

4.1: What does it say now?

X	Y	U	V	W	P	RHS	
$1/5$	0	$\textcircled{1}$	0	$-1/25$	0	4	Land
2	0	0	$\textcircled{1}$	$-12/5$	0	240	Capital
$4/5$	$\textcircled{1}$	0	0	$1/25$	0	96	Crop B
10	0	0	0	8	$\textcircled{1}$	19200	Profit

- First row says $U = 4 - (1/5)X + (1/25)W$
- Last row says $P = 19200 - 10X - 8W$
- X and W are free, what should they be?
- Every acre of crop A we plant costs us \$10!

4.1: Summary

X	Y	U	V	W	P	RHS	
$1/5$	0	①	0	$-1/25$	0	4	Land
2	0	0	①	$-12/5$	0	240	Capital
$4/5$	①	0	0	$1/25$	0	96	Crop B
10	0	0	0	8	①	19200	Profit

- Our free variables are $X = 0$ and $W = 0$
Plant no acres of crop A, and use all available labor
- First row says $U = 4 - (1/5)X + (1/25)W = 4$
Leave 4 acres fallow
- Second row says $V = 240 - 2X + (12/5)W = 240$
Leave \$240 unspent
- Third row says $Y = 96 - (4/5)X - (1/25)W = 96$
Plant 96 acres of crop B
- Last row says $P = 19200 - 10X - 8W = 19200$
Profit is \$19,200

4.1: The key was choosing the right pivot

- How did we know 25 was a good pivot?
- In $U = 100 - X - Y$, if we make Y too big, U goes negative
- “too big” is complicated
- If a variable is free, the only safe bet for its value is 0
- But we had $P = 150X + 200Y$, $Y = 0$ is cowardly
- We needed to make Y a pivot, rather than Y being free
- So we want a pivot in the Y column.

4.1: Choosing the pivot

- Choose the pivot column first:
any column with a negative number at the bottom is OK
“Leftmost” and “Most negative” are reasonable strategies
- Which row?
- Each row has a maximum allowed Y :
 $U = 100 - X - Y$ allows $Y \leq 100$
 $V = 6000 - 50X - 60Y$ allows $Y \leq 100$
 $W = 2400 - 20X - 25Y$ allows $Y \leq 2400/25 = 96$
- The W row is most restrictive, so we use it
- After you pick the pivot column, choose the pivot row by computing these ratios

Choose the smallest non-negative ratio

4.1: Once is not enough

- Usually a single pivot change is not enough
- The bottom row may still have negatives
- Just choose a pivot again, and repeat
- Make sure the right-hand-sides are always non-negative
- If they are negative, problem is harder or you've made a mistake
- See the silly webpage
<http://www.ms.uky.edu/~jack/2011-01-MA162/ch4.html>