

# MA162: Finite mathematics

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March 23, 2011

## SCHEDULE:

- HW C1 is due Monday, Mar 28th, 2011.
- HW C2 is due Monday, Apr 4th, 2011.
- HW C3 is due Sunday, Apr 10th, 2011.
- Exam 3 is Monday, Apr 11th, 5:00pm-7:00pm.
- Alternate exam form (again) due Thursday, Mar 31, 2011.

Today we will cover 5.3: amortized loans.

We will be using calculators today.

# Exam 3 breakdown

- Chapter 5, Interest and the Time Value of Money
  - Simple interest  
short term, interest not reinvested
  - Compound interest  
one payment, interest reinvested
  - Sinking funds  
recurring payments, big money in the future
  - Amortized loans  
recurring payments, big money in the present
- Chapter 6, Counting
  - Inclusion exclusion
  - Inclusion exclusion
  - Multiplication principle
  - Permutations and combinations



## 5.3: Buying annuities

- How much would you pay today for an annuity paying you back \$100 per month for 12 months?
- No more than \$1200 for sure, if you had \$1200 you could just pay yourself
- Let's try to find the right price for such a cash flow
- What if you didn't need the money?  
You could deposit it each month into your savings account.

Earning 1% interest per year, compounded monthly

## 5.2: Annuity reminder

- Remember how to calculate the future value of annuity:

$$A = R((1 + i)^n - 1)/(i)$$

$$R = \$100$$

$$i = 0.01/12$$

$$n = 12$$

$$A = \$100((1 + 0.01/12)^{12} - 1)/(0.01/12)$$

$$A = \$1205.52$$

- How much would we need to put in the bank to have \$1205.52 at the end of the year?

## 5.1: Compound interest reminder

- Remember how to find the present value of future money in a savings account:

$$A = P(1 + i)^n$$

$$P = ?$$

$$i = 0.01/12$$

$$n = 12$$

$$A = \$1205.52$$

$$\$1205.52 = (P)((1 + 0.01/12) \wedge 12)$$

$$\$1205.52 = (P)(1.010045957)$$

$$P = \$1205.52 / 1.010045957 = \$1193.53$$

- \$1193.53 in the bank now, gives \$1205.52 in the bank in a year

## 5.3: Pricing annuities using present value

- If you had \$1193.53 and just put it in the bank now, you'd end up with  $\$1193.53(1 + 1\%/12)^{12} = \$1205.52$  anyways
- If you were just concerned with how much you had in the bank at the end, then you would have no preference between \$1193.53 up front and \$100 each month.
- In other words, the **present value** of the \$100 each month for a year is \$1193.53 because both of those have the same **future value**
- What if you do need the money each month?  
Is \$1193.53 still the right price?

## 5.3: Making your own annuity (endowment)

- What would happen if you put \$1193.53 in the bank, and withdrew \$100 each month?

Month	Bank	Month	Bank
1	1094.52	7	498.76
2	995.44	8	399.17
3	896.27	9	299.51
4	797.01	10	199.76
5	697.68	11	99.92
6	598.26	12	0.00

- At the end of the year, you'd have \$0.00 in the bank, but you would not be overdrawn.
- \$1193.53 **now** gets you \$100 per month for a year

## 5.3: Pricing an annuity

- To price an annuity using our old formulas:
- Find the future value  $A = R((1 + i)^n - 1)/(i)$
- Find the present value by solving  $A = P(1 + i)^n$

$$P = A/((1 + i)^n)$$

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- If you like new formulas, the book divides the  $(1 + i)^n$  using algebra:

$$P = R \left( 1 - (1 + i)^{(-n)} \right) / (i)$$

## 5.3: Perspective

- Bobby Jo borrows \$100 per month from Hank at 1% interest, compounded monthly
  - Hank thinks of Bobby Jo as a savings account
  - Hank expects \$1205.52 in his account at the end of the year
  - Bobby Jo owes Hank \$1205.52 at the end of the year
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- What if the bank called you up and wanted to buy an annuity?
  - What if Hank wants Bobby Jo to pay in advance?  
How much does Bobby Jo owe him up front?

## 5.3: Amortized loan

- Most people don't say "the bank purchased an annuity from me"
- "I owe the bank money every month, because they gave me a loan"
- So the bank gives you \$1193.53 and expects 1% interest
- You give the bank \$100 back at the end of the month

- You owe:

$$\begin{aligned} & \$1193.53 + (1\%/12 \text{ of it}) - \$100 \\ & = \$1193.53 + \$0.99 - \$100 \\ & = \$1094.52 \end{aligned}$$

Month	Debt	Month	Debt
1	1094.52	7	498.76
2	995.44	8	399.17
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- Amortized loans are just present values of annuities

## 5.3: Finding the time

- If you owe \$1000 at 12% interest compounded monthly and pay back \$20 per month, how long does it take to pay it off?
- After one month, you owe  $\$1000 + \$10$  interest -  $\$20$  payment, a total of \$990
- So each month the debt goes down by a net \$10?  
Should take 99 more months, or a little more than 8 years.
- After two months, you owe  $\$990 + \$9.90$  interest -  $\$20$  payment, a total of \$979.90
- Now it went down by \$10.10!  
Should take  $\$979.90 / \$10.10 \approx 97$  months  
After one month of paying, we estimate two months fewer  
How many is it really?

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After one month of paying, we estimate two months fewer  
**How many is it really?**

## 5.3: Finding the time

- The debt is paid once the future value of the annuity is equal to the future value of the debt

- Annuity:

$$A = R((1 + i)^n - 1)/(i)$$

$$R = \$20$$

$$i = 0.12/12 = 0.01$$

$$n = ?$$

$$A = \dots$$

- Debt:

$$A = P(1 + i)^n$$

$$P = \$1000$$

$$i = 0.01$$

$$n = ?$$

$$A = \$1000(1.01)^n$$

- So solve:

$$\$20(1.01^n - 1)/0.01 = \$1000(1.01)^n$$

## 5.3: Algebra

Need to solve:

$$\$20(1.01^n - 1)/0.01 = \$1000(1.01)^n$$

divide both sides by \$1000 and notice  $\$20/0.01/\$1000 = 2$ :

$$2(1.01^n - 1) = 1.01^n$$

distribute:

$$2(1.01^n) - 2 = 1.01^n$$

subtract  $1.01^n$  from both sides, add 2 to both sides:

$$1.01^n = 2$$

Now what?

## 5.3: Logarithms

- To solve:

$$1.01^n = 2$$

- Take **logarithms** of both sides:

$$(n)(\log(1.01)) = \log(2)$$

- $\log(1.01)$  is just a number (some might say 0.004321373783)
- Divide both sides by  $\log(1.01)$  to get:

$$n = \log(2) / \log(1.01) \approx 69.66 \approx 70$$

- $n = 70$  months
- Monthly payments are worth the same as the debt after 70 months