MA162: Finite mathematics

Jack Schmidt

University of Kentucky

October 24, 2011

SCHEDULE:

- HW 5.1-5.3 is due Friday, Oct 28th, 2011.
- HW 6A is due Friday, Nov 4th, 2011.
- HW 6B is due Wednesday, Nov 9th, 2011.
- HW 6C is due Friday, Nov 11th, 2011. (Ch 6 is half easy and half crazy; start now)
- Exam 3 is Monday, Nov 14th, 5:00pm-7:00pm in CB106.

Today we will cover 5.2: annuities.

Exam 3 breakdown

- Chapter 5, Interest and the Time Value of Money
  - Simple interest
  - Compound interest
  - Sinking funds
  - Amortized loans

- Chapter 6, Counting
  - Inclusion exclusion
  - Inclusion exclusion
  - Multiplication principle
  - Permutations and combinations

5.2: Annuities

- "Annuity" can refer to a wide variety of financial instruments, often associated with retirement
- In this class, it is simply a steady flow of cash into an interest bearing account
- For instance, if you put \$100 at the end of every month into your savings account, earning 1% interest, then
- After 1 month: you have the original \$100
- After 2 months you have:
  \$100.00 from the first deposit,
  \$ 0.08 interest on the first deposit, and
  \$100.00 from the second deposit,
  \$200.08 **Total**

5.2: Watching it grow

- After three months, you have:
  \$200.08 from last month,
  \$ 0.17 interest on last month's balance, and
  <u>\$100.00</u> from the third deposit,
  \$300.25 **Total**

- After four months, you have:
  \$300.25 from last month,
  \$ 0.25 interest on last month's balance, and
  <u>\$100.00</u> from the fourth deposit,
  \$400.50 **Total**

- How about after a year? Two years? Fairly tedious this way.

5.2: Formula

$$A = R((1+i)^n - 1)/i$$

- where the **Recurring payment** is how much is deposited at the end of each period, like \$100
- the **interest rate** per period, like $1\%/12$
- the **number of periods**, like four months
- the **accumulated amount**, like

$$A = \$100((1+1\%/12)^4 - 1)/(1\%/12) = \$400.50$$

$$A = 100 \star ((1+0.01/12) \wedge 4 - 1)/(0.01/12) = 400.5001200$$

5.2: Examples of formula

$$A = R((1+i)^n - 1)/i$$

- After one year of investing \$100 at the end of every month at a 1% (nominal yearly) interest rate:

$$\begin{aligned} R &= \$100 \\ i &= 1\%/12 \\ &\approx 0.00833333 \\ n &= 12 \text{ months} \\ A &= \$100((1+1\%/12)^{12} - 1)/(1\%/12) \approx \$1205.52 \end{aligned}$$

- After two years of investing \$100 at the end of every month at a 1% (nominal yearly) interest rate:

$$\begin{aligned} R &= \$100 \\ i &= 1\%/12 \\ &\approx 0.00833333 \\ n &= 24 \text{ months} \\ A &= \$100((1+1\%/12)^{24} - 1)/(1\%/12) \approx \$2423.14 \end{aligned}$$

5.2: Retirement example

- UK employees aged 30 or over must contribute 5% of their salary each month to a retirement plan, which UK doubles, a total of 15%
- If a UK employee makes \$35k and retires at age 65 and manages to earn a steady 8% interest rate, then they retire with:

$$\begin{aligned} R &= (\$35000)(15\%)/12 = \$437.50 \\ i &= 8\%/12 \\ n &= (35)(12) = 420 \text{ months} \\ A &= \$437.50((1+8\%/12)^{420}-1)/(8\%/12) \approx \$1,003,573.59 \end{aligned}$$

- If a UK employee makes \$70k and retires at age 65 and manages to earn a steady 8% interest rate, then they retire with:

$$\begin{aligned} R &= \$875 \\ i &= 8\%/12 \\ n &= (35)(12) = 420 \text{ months} \\ A &= \$875((1+8\%/12)^{420}-1)/(8\%/12) \approx \$2,007,147.18 \end{aligned}$$

5.2: Sinking fund example

- Businesses can often predict future expenses; our building needs a new water boiler (\$80k) after this one breaks
- We set aside a little each month so that we have it when we need it
- If we can get 3% interest in low-risk investments and expect the boiler to fail in 5 years, we need to invest R per month:

$$
\begin{array}{rl}
\text{A} & = R((1+i)^n - 1)/(i) \\
\hline
\text{R} & = ? \\
\text{i} & = 3\%/12 \\
\text{n} & = (5)(12) = 60 \text{ months} \\
\text{A} & = \$80000 \\
\$80000 & = R((1+3\%/12)^{60} - 1)/(3\%/12) \\
\$80000 & = R(64.64671280) \\
\text{R} & = \$80000/64.64671280 = \$1237.50
\end{array}
$$

5.2: Sinking fund versus one-time-investment

- Maybe we don't want to pay a little each month
- Maybe we just want to invest a whole bunch now and cash in later

$$\begin{array}{rl}
\text{A} & = P(1+i)^n \\
\hline
\text{P} & = ? \\
\text{i} & = 3\%/12 \\
\text{n} & = (5)(12) = 60 \text{ months} \\
\text{A} & = \$80000 \\
\$8000 & = P(1+3\%/12)^{60} \\
\$8000 & = P(1.161616782) \\
\text{P} & = \$80000/1.161616782 = \$68869.53
\end{array}$$

- Less total money we invested for same future value
- But we need that \$68k NOW, not \$1.2k at a time

5.2: Why does the formula work?

- After one month you have \$100
- The next month you add a fresh \$100 and (1+i) times your previous month
  $\$100 + \$100 \cdot (1+i)$
- The next month you add a fresh \$100 and (1+i) times your previous month
  $\$100 + (\$100 + \$100 \cdot (1+i)) \cdot (1+i)$
  $\$100 + \$100 \cdot (1+i) + \$100 \cdot (1+i)^2$
- The next month you add a fresh \$100 and (1+i) times your previous month
  $\$100 + (\$100 + (\$100 + \$100 \cdot (1+i)) \cdot (1+i)) \cdot (1+i)$
  $\$100 + \$100 \cdot (1+i) + \$100 \cdot (1+i)^2 + \$100 \cdot (1+i)^3$

5.2: Trick for summations

- After n months you have added up n things:

$$A = \$100 + \$100 \cdot (1+i) + \cdots + \$100 \cdot (1+i)^{n-1}$$

- Let's try a trick. What happens if I let the money ride for a month? It earns interest, so I have $A \cdot (1+i)$ in the bank.

- How much more is that? Well $A \cdot (1+i) - A = Ai$ is not tricky.

- But multiply it out before doing the subtraction:

$$\begin{array}{rcccccccccccc} & A \cdot (1+i) & = & & & \$100 \cdot (1+i) & + & \ldots & + & \$100 \cdot (1+i)^{n-1} & + & \$100 \cdot (1+i)^n \\ - & A & = & \$100 & + & \$100 \cdot (1+i) & + & \ldots & + & \$100 \cdot (1+i)^{n-1} & & \\ \hline & Ai & = & -\$100 & & & & & & & + & \$100 \cdot (1+i)^n \end{array}$$

- So $Ai = \$100 \cdot ((1+i)^n - 1)$ and we can solve for A:

$$A = \$100 \frac{(1+i)^n - 1}{i}$$

5.2: Time value of money and total payout

- How much would you pay me for (the promise of) \$100 in a year?
- Future money is not worth as much as money right now
  "A bird in the hand, is worth two in the bush" posits an interest rate of 100%
- Present value of future money **depreciates** the value of future money by comparing it to present money invested in the bank now
- **Total payout** is a popular measure of a financial instrument, but it mixes present money, with in-a-little-while money, with future money
- Total payout of an annuity is just the total amount you put in the savings account (or the total amount you borrowed each month)

5.2: Summary

- Today we learned about **annuities**, **present value**, **future value**, and **total payout**
    - Future value of annuity, paying out n times at per-period interest rate i
    $$A = R\frac{(1+i)^n - 1}{i}$$
    - Present value of annuity is just future value divided by $(1+i)^n$
    - Total payout is just nR, n payments of R each
- You are now ready to complete HWC1 (and should have done 3 or 4 problems)
- Make sure to take advantage of office hours

5.3: Buying annuities

- How much would you pay today for an annuity paying you back \$100 per month for 12 months?
- No more than \$1200 for sure,
  if you had \$1200 you could just pay yourself
- Let's try to find the right price for such a cash flow
- What if you didn't need the money?
  You could deposit it each month into your savings account.
- We already calculated that you end up with \$1205.52 if you do that
- How much would you pay today for \$1205.52 in the bank a year from now?

5.3: Pricing annuities

- If you had \$1193.53 and just put it in the bank now, you'd end up with $\$1193.53(1 + 1\%/12)^{12} = \$1205.52$ anyways
- If you were just concerned with how much you had in the bank at the end, then you would have no preference between \$1193.53 up front and \$100 each month.
- In other words, the **present value** of the \$100 each month for a year is \$1193.53 because both of those have the same **future value**
- What if you do need the money each month?
  Is \$1193.53 still the right price?

5.3: Pricing annuities again

- What would happen if you put \$1193.53 in the bank, and withdrew \$100 each month?
- At the end of the year, you'd have \$0.00 in the bank, but you would not be overdrawn.
- Why is that? Imagine borrowing money from your friend, \$100 every month and not paying them back
- They know you pretty well, so they insisted on 1% interest, compounded monthly
- How much do you owe them at the end?
- Well from their point of view, they gave their money to you, just like putting it in a savings account
- The bank would have owed them \$1205.52, so you owe them \$1205.52. Now imagine your savings account is your friend.