MA162: Finite mathematics

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October 26, 2011

SCHEDULE:

- HW 5.1-5.3 is due Friday, Oct 28th, 2011.
- HW 6A is due Friday, Nov 4th, 2011.
- HW 6B is due Wednesday, Nov 9th, 2011.
- HW 6C is due Friday, Nov 11th, 2011. (Ch 6 is half easy and half crazy; start now)
- Exam 3 is Monday, Nov 14th, 5:00pm-7:00pm in CB106.

Today we will cover 5.3: amortized loans. We will be using calculators today.

Exam 3 breakdown

- Chapter 5, Interest and the Time Value of Money
  - Simple interest
    short term, interest not reinvested
  - Compound interest
    one payment, interest reinvested
  - Sinking funds
    recurring payments, big money in the future
  - Amortized loans
    recurring payments, big money in the present

- Chapter 6, Counting
  - Inclusion exclusion
  - Inclusion exclusion
  - Multiplication principle
  - Permutations and combinations

5.2: Time value of money and total payout

- How much would you pay me for (the promise of) $100 in a year?
- Future money is not worth as much as money right now
  "A bird in the hand, is worth two in the bush" posits an interest rate of 100%
- Present value of future money **depreciates** the value of future money by comparing it to present money invested in the bank now
- **Total payout** is a popular measure of a financial instrument, but it mixes present money, with in-a-little-while money, with future money
- Total payout of an annuity is just the total amount you put in the savings account (or the total amount you borrowed each month)

5.2: Summary

- Monday we learned about **annuities**, **present value**, **future value**, and **total payout**
  - Future value of annuity, paying out n times at per-period interest rate i
    $$A = R\frac{(1+i)^n - 1}{i}$$
  - Present value of annuity is just future value divided by $(1+i)^n$
  - Total payout is just nR, n payments of R each
- You are now ready to complete 5.1 and 5.2 (and should have probably done all of them anyways).
- Now we handle 5.3.

5.3: Buying annuities

- How much would you pay today for an annuity paying you back $100 per month for 12 months?
- No more than $1200 for sure,
  if you had $1200 you could just pay yourself
- Let's try to find the right price for such a cash flow
- What if you didn't need the money?
  You could deposit it each month into your savings account.
- We already calculated that you end up with $1205.52 if you do that
- How much would you pay today for $1205.52 in the bank a year from now?

5.3: Pricing annuities

- If you had \$1193.53 and just put it in the bank now, you'd end up with $\$1193.53(1 + 1\%/12)^{12} = \$1205.52$ anyways

- If you were just concerned with how much you had in the bank at the end, then you would have no preference between \$1193.53 up front and \$100 each month.

- In other words, the **present value** of the \$100 each month for a year is \$1193.53 because both of those have the same **future value**

- What if you do need the money each month?
  Is \$1193.53 still the right price?

5.3: Pricing annuities again

- What would happen if you put \$1193.53 in the bank, and withdrew \$100 each month?
- At the end of the year, you'd have \$0.00 in the bank, but you would not be overdrawn.
- Why is that? Imagine borrowing money from your friend, \$100 every month and not paying them back
- They know you pretty well, so they insisted on 1% interest, compounded monthly
- How much do you owe them at the end?
- Well from their point of view, they gave their money to you, just like putting it in a savings account
- The bank would have owed them \$1205.52, so you owe them \$1205.52. Now imagine your savings account is your friend.

5.3: Buying annuities

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- What if you didn't need the money?
  You could deposit it each month into your savings account.

  Earning 1% interest per year, compounded monthly

5.2: Annuity reminder

- Remember how to calculate the future value of annuity:

$$
\begin{aligned}
\text{A} &= R((1+i)^n - 1)/(i) \\
\\
\text{R} &= \$100 \\
\text{i} &= 0.01/12 \\
\text{n} &= 12 \\
\\
\text{A} &= \$100((1+0.01/12) \wedge 12 - 1)/(0.01/12) \\
\text{A} &= \$1205.52
\end{aligned}
$$

- How much would we need to put in the bank to have \$1205.52 at the end of the year?

5.1: Compound interest reminder

- Remember how to find the present value of future money in a savings account:

$$
\begin{aligned}
\text{A} &= P(1+i)^n \\
\\
\text{P} &= ? \\
\text{i} &= 0.01/12 \\
\text{n} &= 12 \\
\text{A} &= \$1205.52 \\
\\
\$1205.52 &= \text{(P)}((1+0.01/12) \wedge 12) \\
\$1205.52 &= \text{(P)}(1.010045957) \\
\text{P} &= \$1205.52 \ / \ 1.010045957 = \$1193.53
\end{aligned}
$$

- \$1193.53 in the bank now, gives \$1205.52 in the bank in a year

5.3: Pricing annuities using present value

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- What if you do need the money each month?
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5.3: Making your own annuity (endowment)

- What would happen if you put \$1193.53 in the bank, and withdrew \$100 each month?

Month	Bank	Month	Bank
1	1094.52	7	498.76
2	995.44	8	399.17
3	896.27	9	299.51
4	797.01	10	199.76
5	697.68	11	99.92
6	598.26	12	0.00

- At the end of the year, you'd have \$0.00 in the bank, but you would not be overdrawn.

- \$1193.53 **now** gets you \$100 per month for a year

5.3: Pricing an annuity

- To price an annuity using our old formulas:
- Find the future value $A = R((1+i)^n - 1)/(i)$
- Find the present value by solving $A = P(1+i)^n$

$$P = A/((1+i)^n)$$

- If you like new formulas, the book divides the $(1+i)^n$ using algebra:

$$P = R\left(1 - (1+i)^{(-n)}\right)/(i)$$

5.3: Perspective

- Bobby Jo borrows \$100 per month from Hank at 1% interest, compounded monthly
- Hank thinks of Bobby Jo as a savings account
- Hank expects \$1205.52 in his account at the end of the year
- Bobby Jo owes Hank \$1205.52 at the end of the year

- What if the bank called you up and wanted to buy an annuity?
- What if Hank wants Bobby Jo to pay in advance?
  How much does Bobby Jo owe him up front?

5.3: Amortized loan

- Most people don't say "the bank purchased an annuity from me"
- "I owe the bank money every month, because they gave me a loan"
- So the bank gives you \$1193.53 and expects 1% interest
- You give the bank \$100 back at the end of the month
- You owe:
  $\$1193.53 + (1\%/12 \text{ of it}) - \$100$
  $= \$1193.53 + \$0.99 - \$100$
  $= \$1094.52$

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- Amortized loans are just present values of annuities

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- If you owe \$1000 at 12% interest compounded monthly and pay back \$20 per month, how long does it take to pay it off?

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- So each month the debt goes down by a net \$10?
  Should take 99 more months, or a little more than 8 years.

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  Should take \$979.90/\$10.10 $\approx$ 97 months
  After one month of paying, we estimate two months fewer

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  Should take $\$979.90/\$10.10 \approx 97$ months
  After one month of paying, we estimate two months fewer
  How many is it really?

5.3: Finding the time

- The debt is paid once the future value of the annuity is equal to the future value of the debt

- Annuity:

$$\begin{aligned} A &= R((1+i)^n - 1)/(i) \\ \\ R &= \$20 \\ i &= 0.12/12 = 0.01 \\ n &= ? \\ A &= \ldots \end{aligned}$$

  Debt:

$$\begin{aligned} A &= P(1+i)^n \\ \\ P &= \$1000 \\ i &= 0.01 \\ n &= ? \\ A &= \$1000(1.01)^n \end{aligned}$$

- So solve:

$$\$20(1.01^n - 1)/0.01 = \$1000(1.01)^n$$

5.3: Algebra

Need to solve:

$$\$20(1.01^n - 1)/0.01 = \$1000(1.01)^n$$

divide both sides by \$1000 and notice $\$20/0.01/\$1000 = 2$:

$$2(1.01^n - 1) = 1.01^n$$

distribute:

$$2(1.01^n) - 2 = 1.01^n$$

subtract 1.01^n from both sides, add 2 to both sides:

$$1.01^n = 2$$

Now what?

5.3: Logarithms

- To solve:

$$1.01^n = 2$$

- Take **logarithms** of both sides:

$$(n)(\log(1.01)) = \log(2)$$

- $\log(1.01)$ is just a number (some might say 0.004321373783)
- Divide both sides by $\log(1.01)$ to get:

$$n = \log(2)/\log(1.01) \approx 69.66 \approx 70$$

- $n = 70$ months
- Monthly payments are worth the same as the debt after 70 months